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Algebra
This book is dedicated to Veritas Prep’s instructors, whose enthusiasm and experience have contributed mightily to our educational philosophy and our students’ success.

It is also dedicated to the teachers who inspired Veritas Prep’s instructors. The lesson that follows was only made possible by a lifelong love of learning and of undertaking educational challenges; we have teachers around the world to thank for that.

Finally and most importantly, this book is dedicated to our thousands of students, who have taught us more about teaching and learning than they will ever know. And to you, the reader, thank you for adding yourself to that group.

**Personal Dedications**

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CREATING *Think Like the Testmaker*

Creating is the top of the pyramid in Bloom’s Taxonomy. When you have completely mastered the GMAT, you are able to Think Like the Testmaker. You are on top of the pyramid looking down! You don’t just have good content knowledge and lots of practice with GMAT problems; you understand how a problem has been made, what makes it hard, and how to break it down. When you Think Like the Testmaker you can:

1. Quickly recognize what the problem is actually asking,
2. Discover hidden information and manipulate it to make it useful,
3. Recognize and see through trap answers, and
4. Create your own plan of attack for any problem.

APPLYING *Skills Meet Strategy*

What makes the GMAT difficult is not so much the underlying skills and concepts, but rather the way those skills and concepts are tested. On the GMAT, what you know is only as valuable as what you can do with that knowledge. The Veritas Prep curriculum emphasizes learning through challenging problems so that you can:

1. Learn how to combine skills and strategies to effectively solve any GMAT problem,
2. Most effectively utilize the classroom time you spend with a true GMAT expert, and
3. Stay focused and engaged, even after a long day in the office.

REMEMBERING *Skillbuilder*

In order to test higher-level thinking skills, testmakers must have some underlying content from which to create problems. On the GMAT, this content is primarily:

- Math curriculum through the early high school level, and
- Basic grammar skills through the elementary school level.

To succeed on the GMAT you must have a thorough mastery of this content, but many students already have a relatively strong command of this material. For each content area, we have identified all core skills that simply require refreshing and/or memorizing and have put them in our *Skillbuilder* section. By doing this:

1. Students who need to thoroughly review or relearn these core skills can do so at their own pace, and
2. Students who already have a solid command of the underlying content will not become disengaged because of a tedious review of material they’ve already mastered.
As you learned in the Foundations of GMAT Logic lesson, the educational philosophy at Veritas Prep is based on the multi-tiered *Bloom’s Taxonomy of Educational Objectives*, which classifies different orders of thinking in terms of understanding and complexity.

To achieve a high score on the GMAT, it is essential that you understand the test from the top of the pyramid. On the pages that follow, you will learn specifically how to achieve that goal and how this lesson in particular relates to the *Veritas Prep Pyramid*. 
How This Book Is Structured

Our Curriculum Is Designed to Maximize the Value of Your Time

The Veritas Prep Teaching Philosophy: Learning by Doing

Business schools have long featured the Case Method of education, providing students with real-world problems to solve by applying the frameworks they have studied. The Veritas Prep Learning by Doing method is similar. In class, you will spend your time applying skills and concepts to challenging GMAT problems, at the same time reviewing and better understanding core skills while focusing your attention on application and strategy. The Case Method in business school maximizes student engagement and develops higher-order thinking skills, because students must apply and create, not just remember. Similarly, the Learning by Doing philosophy maximizes the value of your study time, forcing you to engage with difficult questions and develop top-of-the-pyramid reasoning ability.

An important note on Learning by Doing: In business school, your goal with a business case is not to simply master the details of a particular company’s historical situation, but rather to develop broader understanding of how to apply frameworks to real situations. In this course, you should be certain to reflect on each question not simply through that narrow lens (Did you answer correctly? What key word made the difference?), but rather as an example of larger GMAT strategy (How could the exam bait you with a similar trap? How deeply do you need to understand the content to solve this genre of problem more efficiently?).
How This Book Is Structured

As you learned in the Foundations of GMAT Logic lesson, there are important recurring themes that you will see in most GMAT problems:

**THINK LIKE THE TESTMAKER**

- Abstraction
- Reverse-Engineering
- Large or Awkward Numbers
- Exploiting Common Mistakes
- Selling the Wrong Answer and Hiding the Correct Answer
- Misdirection
- Content-Specific Themes

**SKILLS MEET STRATEGY**

- Guiding Principles
- Problem-Solving Strategies
- Leveraging Assets

REMEMBER: Don’t mistake activity for achievement! Focus on recurring themes, not just underlying content.
Each book in the Veritas Prep curriculum contains four distinct sections:

1. **Skillbuilder.** We strongly suggest that you **complete each Skillbuilder lesson before class** at your own pace, and return to the Skillbuilder when you recognize a content deficiency through practice tests and GMAT homework problem sets.

   The Skillbuilder section will:
   - Cover content that is **vital to your success on the GMAT**, but is best learned at your own pace outside the classroom.
   - Allow you to **review and/or relearn** the skills, facts, formulas, and content of the GMAT. Each student will have his own set of skills that are “rusty” or even brand-new, and will find other items that come back quickly.
   - **Vary in length** significantly for each book, based on the number of underlying concepts. (For instance, the Advanced Verbal lesson does not have a Skillbuilder because you are already building on the concepts introduced in three previous lessons.)

2. **Lesson.** The lessons are designed to provide students with maximum value added from an instructor by:

   - Doing in-class problems together (**Learning by Doing**), and
   - Analyzing those problems for the recurring takeaways.

   With each problem, there will be a detailed explanation that will help you understand how the problem is testing a particular concept or series of concepts, what makes the problem hard, and what underlying skills are required to solve it.

   When relevant, there will be particular boxes for **Think Like the Testmaker**, **Skills Meet Strategy**, and **Skillbuilder** when you should be focused on particular aspects of how the question is made or how the underlying content is being tested.

   **NOTE:** When doing in-class and homework problems, you should **do your work below the problem**, and you **should not circle the answer** on the actual question (just note it on the bottom of the page). That way, if you want to redo problems, you can simply cover up your work and proceed as if you had never done it.
3. **You Oughta Know.** The *You Oughta Know* sections will round out each lesson and cover:

- Obscure topics that arise infrequently.
- More advanced topics that are not common on the GMAT but do get tested.

While these uncommon content areas do not warrant in-class time, we believe you should have some exposure to these topics before taking the GMAT. Therefore you should **complete these sections before moving to the homework problems.** As with the Skillbuilders, the length of these will vary depending on their importance.

4. **Homework Problems.** In many ways, the homework problems are **the most important part of each book.** After refreshing core content in the Skillbuilder and then applying that knowledge in the lesson, you must reinforce your understanding with more problems.

Each question is accompanied by a **detailed explanation** in your online student account, as well as a quick-reference answer key on the last page. A majority of questions are above the 50th percentile in difficulty, and they are arranged in approximate order of difficulty (easiest to most difficult). By completing all of the homework problems, you will learn all of the different iterations of how concepts and skills are tested on the GMAT.

Homework problems are designed to be challenging, so do not despair if you are answering questions incorrectly as you practice! Your goal should be to learn from every mistake. Students can miss a significant percentage of questions in each book and still score extremely high on the GMAT, provided that they learn from each problem. Embrace the challenge of hard problems and the notion that every mistake you make in practice is one that you will know to avoid on the GMAT when every question counts.
Traditionally, about 25% of GMAT quantitative problems have been deemed “algebra problems,” but the percentage of problems that require you to use algebra and algebraic logic is much higher. As you will see throughout the lesson, GMAT Algebra can be summarized in one phrase-slash-movie-title: An Inconvenient Truth. It involves the art of taking a true mathematical statement and leveraging it to make it useful for the purposes of solving a problem. For example, a Data Sufficiency question might ask (here just considering one statement):

If \( x + z \neq 0 \), what is the value of \( \frac{y}{x + z} \)?

(1) \( \frac{x - y}{z} = -1 \)

Here, the statement is true (it’s given as a fact), but it’s inconvenient. Using algebra skills and the knowledge that you often need to manipulate algebra to rephrase that truth in a more convenient fashion, you can use the statement to create an even more valuable equation:

Multiply both sides by \( z \) to get: \( x - y = -z \)

Add \( y \) and \( z \) to both sides to get: \( x + z = y \)

And then look back at the question. If the numerator equals the denominator as you’ve proven by manipulating the statement then the fraction equals 1. Statement (1) is sufficient, but it requires some work to prove that. In other words, the GMAT offered you an inconvenient-but-sufficient statement, and challenged you to use algebra to better assess what the statement said.

In this Skillbuilder, you will review the major algebraic rules and principles so that you have the skills necessary to leverage algebraic information in questions like the above. Remember this: The competitive standard for algebra on the GMAT is high, so you should work to become efficient at recognizing common algebraic structures and at manipulating algebraic statements. Algebra is arguably the most important family of skills tested on the GMAT, so we strongly encourage you to take advantage of this Skillbuilder to make algebra second nature again.
The Algebra Toolkit

As you saw from the demonstration on the previous page, GMAT Algebra is as much about “what you can do” as it is about “what you know.” Accordingly, we’ve organized our coverage of algebra rules and principles along the lines of this Algebra Toolkit, which demonstrates what you are able to do with algebra as you attempt to make statements more convenient and to solve for variables.

With algebra, you can:

• Multiply by 1—use the same numerator and denominator to simplify fractions.
• Combine Like Terms/Factor—rearrange terms to simplify expressions.
• Do the Same to Both Sides—perform operations in an equation to isolate variables.
• Eliminate Variables—combine equations to reduce the number of variables.

This lesson will also cover special principles for:

• Exponents/Roots.
• Inequalities and Absolute Values.
• Quadratics and Multiple Solutions.

Throughout the lesson, you should focus your efforts on not merely knowing the algebra rules, but recognizing when and how to employ them to simplify or rearrange expressions. Take advantage of the many drills in this lesson to practice facility with algebra so that you are comfortable performing these operations efficiently and effectively on test day.
Multiply by 1

On the GMAT, many of the “inconvenient” expressions and equations that you encounter will involve fractions and variables. Almost always, the simplest forms of these expressions and equations will be found either through common denominators or through the elimination of denominators. To do this, strategically multiply by 1 to either make denominators common or to eliminate them altogether. Consider the following expression:

\[ \frac{4a}{3} + a \]

In order to simplify this expression, you need to create a common denominator by multiplying the lone a term by \( \frac{3}{3} \), a manipulation that is perfectly valid because \( \frac{3}{3} \) is just another way to express 1, and multiplying a number by 1 does not change its value. So the expression becomes:

\[ \frac{4a}{3} + \frac{3a}{3} \]

From this, you can now add the like terms to get:

\[ \frac{7a}{3} \]

Finding common denominators is essential for adding/subtracting fractions, whether you’re dealing with strictly variables, strictly numbers, or a combination of the two. Note, for instance, that the process above—multiplying by a strategically chosen form of 1—is the same that you use to solve the following problem:

What is \( \frac{5}{9} + \frac{1}{6} \)?
A common denominator for these fractions is 18, so you should multiply the first term by $\frac{2}{2}$ and the second term by $\frac{3}{3}$ to get:

$$\frac{2 \cdot 5}{2 \cdot 9} + \frac{3 \cdot 1}{3 \cdot 6}$$

Carrying out the multiplication and then adding the now-like terms gives us:

$$\frac{10}{18} + \frac{3}{18} = \frac{13}{18}$$

You should be comfortable multiplying by 1. It’s the strategic use of this concept that can make your GMAT experience much easier. Consider this:

What is $\frac{\left(\frac{7a}{3} - \frac{5a}{9}\right)}{\frac{a}{9}}$ ?

While this problem may look at first glance to require multiple steps, you can strategically lighten the load by multiplying by $\frac{9}{9}$, recognizing that doing so eliminates all three denominators (3, 9, and 9) within the larger fraction:

$$\frac{\left(\frac{7a}{3} - \frac{5a}{9}\right)}{\frac{a}{9}} = \frac{(7a)(3) - (5a)}{a} = \frac{21a - 5a}{a} = \frac{16a}{a} = 16$$

Multiplying by 1 is an extremely helpful strategy in many algebra problems, particularly those that involve multiple denominators. One of your foremost goals when simplifying problems should be to get rid of denominators, as the above example shows.
Multiply by 1 Drills

1. What is $\frac{4x}{3}$?

2. What is $\frac{(5x + 3x)}{12 + 4}$?

3. What is $\frac{y + 3y}{2}$?

4. Simplify: $n + \frac{3n}{2} - \frac{7n}{3}$.

5. What is $\frac{x + \frac{5x}{6}}{1}$?
Solutions to Multiply by 1 Drills

1. Multiply the top and bottom by 9: \[ \frac{\frac{4x}{3}}{\frac{5x}{9}} \cdot \frac{\frac{9}{9}}{\frac{9}{9}} = \frac{12x}{5x} = \frac{12}{5} \]

2. Multiply the top and bottom by 12: \[ \frac{\frac{5x}{12} + \frac{3x}{4}}{\frac{x}{6}} \cdot \frac{\frac{12}{12}}{\frac{12}{12}} = \frac{5x + 9x}{2x} = \frac{14x}{2x} = 7 \]

3. Multiply the top and bottom by 4: \[ \frac{\frac{y + \frac{3y}{2}}{\frac{1}{4}}}{\frac{\frac{1}{4}}{\frac{1}{4}}} = \frac{4y + 6y}{1} = 10y \]

4. Multiply all terms by \( \frac{6}{6} \): \[ \frac{\frac{6n + \frac{6(3n)}{2}}{\frac{6}{6}}}{\frac{\frac{6(7n)}{3}}{\frac{6}{6}}} = \frac{6n + 9n - 14n}{6} = \frac{n}{6} \]

5. Multiply the top and bottom by 12: \[ \frac{\frac{x + \frac{5x}{6}}{\frac{1}{4}}}{\frac{\frac{12}{12}}{\frac{3}{3}}} = \frac{12x + 10x}{3} = \frac{22x}{3} \]
**Combine Like Terms/Factor**

Often, algebra is made much simpler by simply rearranging the terms so that like terms are placed together or combined. For example, if a question presented you with this statement:

\[ 2x + 3y + 5x + 4y \]

You can rearrange the order so that like terms are together:

\[ 2x + 5x + 3y + 4y \]

Then combine the like terms to reduce the number of overall terms:

\[ 7x + 7y \]

Strategy becomes important when you must use that type of manipulation on a Data Sufficiency question such as (again just considering one statement) this:

What is the value of \( x + y \)?

(1) \( 2x + 3y + 5x + 4y = 21 \)

In this case, you know that the statement also says, as you derived above:

\[ 7x + 7y = 21 \]

Here we'll want to take one more step and factor the "like term" of 7:

\[ 7(x + y) = 21 \]

Now you can divide both sides by 7 to find that \( x + y = 3 \) (making the statement sufficient).

Strategically combining like terms and factoring should be integral parts of your algebraic arsenal. The following pages include a review of what can or can’t be combined; following that is a series of drills to build your familiarity with these techniques.
**Addition and Subtraction**

One cannot combine unlike terms in addition and subtraction. That is, one cannot combine:

- An unknown variable with a number, as in x and 1. The expressions x + 1 and x – 1 cannot be simplified further.

- Two different variables. The expressions x + y and x – y cannot be simplified further.

- Two like variables of different powers, such as x and x². Neither the expression x + x² nor the expression x – x² can be condensed into one term (although, as you will see, they could be factored).

One can combine like terms. Do so by adding or subtracting coefficients:

- 3x + 5x = 8x

- 9y – y = 8y

**Examples**

4x + 3 + 2x – 1 = 6x + 2

x² + 2y + 2y – 3 – y² + 4 – x² – x = –y² + 4y – x + 1
**Multiplication**

When you multiply two different variables or a single number and a variable, neither term disappears, nor do new variables or numbers appear. The convention is simply to place them next to each other: \(a \cdot b = ab\).

Remember that, based on the commutative property, the order of the terms is not important. However, it is standard when writing simplified expressions to write coefficients first, then variables in alphabetical order. Since the GMAT nearly always adheres to this convention in the answer choices, you will recognize correct answers more efficiently if you also develop the habit of writing simplified answers in this standard order.

**Examples**

\[5 \cdot x = 5x\]
\[x \cdot y \cdot z = xyz\]
\[x^2 \cdot 4 \cdot z = 4x^2z\]
\[2 \cdot z \cdot x \cdot y \cdot 3 = 6xyz\]

Multiplying two like variables results in a single variable raised to the power of the sum of the powers of the variables being multiplied: \(a \cdot a^2 = a^{1+2} = a^3\).

**Examples**

\[x \cdot x \cdot 5 = 5x^2\]
\[x \cdot x \cdot y \cdot x \cdot z \cdot y = x^3y^2z\]
\[x^3 \cdot x^2 = x^5\]
\[5 \cdot x^4 \cdot 2 \cdot y \cdot y^2 \cdot x = 10x^5y^3\]
Distributive Property

When multiplication is acting on an expression in parentheses, the multiplication must be distributed to each term within the parentheses:

\[2(x+y) = 2x + 2y\]

Distributing multiplication allows you to eliminate parentheses and to “free” a variable that might be otherwise inaccessible because it’s locked within. For example, if a question were to ask you to solve for \(x\) and provide the equation:

\[3(x – 7) = 2(x – 6)\]

You would distribute the multiplication so that you can eventually combine the \(x\) terms:

\[3x – 21 = 2x – 12\]

Subtract 2\(x\) and add 21 to both sides to get:

\[x = 9\]

Examples

\[A \cdot (B + C) = A \cdot B + A \cdot C = AB + AC\]

\[3 \cdot (12 + 7) = (3 \cdot 12) + (3 \cdot 7) = 36 + 21 = 57\]

\[5(x + y) = 5x + 5y\]

\[6(3 + x) = 6 \cdot 3 + 6x = 18 + 6x\]
Factoring: The Distributive Property in Reverse

On the GMAT, the distributive property is even more useful in reverse as a way to factor out common terms. Remember from the Arithmetic lesson: The GMAT frequently tests factors, multiples, and divisibility, so your ability to turn addition/subtraction into multiplication/division will be paramount. Just as:

\[ A(B + C) = AB + AC \]

So:

\[ AB + AC = A(B + C) \]

This can be extremely important when you’re attacking problems that require or involve multiplication and division. For example:

\[ \frac{x - (3x + 2y)}{x + y} = \]

First remove parentheses by distributing the negative in the numerator:

\[ \frac{x - 3x - 2y}{x + y} = \frac{-2x - 2y}{x + y} \]

Next, factor the common \(-2\) out in the numerator to create a situation in which you can divide out the numerator and denominator. \(-2x - 2y\) is the same as \(-2x + (-2y)\), so you can factor out the common \(-2\):

\[ -2(x + y) \]
\[ x + y \]

The common \((x + y)\) terms will cancel, leaving the result of \(-2\).

**NOTE:** It’s no coincidence that you spent a lot of time in the Arithmetic lesson on factors and that one of the most critical concepts in Algebra is also factoring. Using the concept of factoring to simplify expressions is one of the truly core concepts to the GMAT. When in doubt, look to see if you can factor your way out of a jam.

**Examples**

\[ 3x + 9y = 3(x + 3y) \]
\[ 12a - 16b = 4(3a - 4b) \]
\[ 2a + 6b - 4c = 2(a + 3b - 2c) \]
\[ 3^8 - 3^7 = 3^7(3 - 1) = 2(3^7) \]
Combine Like Terms/Factor Drills

1. Simplify \(2(x + y) - (x - y)\)
2. Simplify \(x(xy)^2\)
3. Simplify \(4(3x + 2y - 18) + 16x - 3y + 14\)
4. Simplify \(yx + xy + xy^2 + (xy)^2\)
5. What is \(\frac{x(x + y) + 2(x + y)}{2 + x}\)?
6. Simplify \(\sqrt{x} + \sqrt{4x}\)
7. Simplify \(2 - (x + 4) + 2x - 4\)
8. What is \(\frac{3 + 9x}{3 + x}\)?
9. If \(x^3 + x^3 + x^3 + x^3 + x^3 = 40\), what is \(x\)?
10. Simplify \(\frac{x + \frac{x}{2} + \frac{x}{4} + \frac{x}{8}}{15}\)
11. Simplify \(\frac{7x + 14y + 28z}{28x + 14y + 7z}\)
12. Simplify \(\frac{58x - (17 + 7x)}{17}\)
13. If \(x + x + x + x = x - 4x\), what is \(x\)?
14. What is \(\frac{x - 7}{7 - x}\)?
15. When the sum of the first 20 multiples of 10 is divided by the sum of the first 20 even numbers, what is the result?

*For the purposes of this drill, assume that no denominator can equal 0. Because division by 0 is prohibited, GMAT questions will preclude that by defining, for example in #5, that “\(x \neq -2\).” Since that notation can become cumbersome in a question stem, for the sake of simplicity here note that no denominator in this drill can equal zero.
Solutions to Combine Like Terms/Factor Drills

1. Simplify $2(x + y) - (x - y)$.
   Distribute multiplication: $2x + 2y - x + y$.
   Combine like terms: $3y + x$.
   **NOTE:** Be careful when you see subtraction outside of parentheses! If you incorrectly answered $x + y$ here, it’s likely because you forgot to multiply the $-y$ in the second parentheses by $-1$. $-(x - y)$ is the same as $-1(x - y) = -1(x) - (-1)(y)$. The two negatives before $y$ multiply to a positive.

2. Simplify $x(xy)^2$.
   Remember that you need to perform calculations within parentheses first, so you first need to turn $(xy)^2$ into $x^2y^2$. $x \cdot x^2y^2 = x^3y^2$.

3. Simplify $4(3x + 2y - 18) + 16x - 3y + 14$.
   Distribute multiplication: $12x + 8y - 72 + 16x - 3y + 14$.
   Then, combine like terms: $28x + 5y - 58$.

4. Simplify $yx + xy + xy^2 + (xy)^2$.
   $yx$ is the same as $xy$ (order doesn’t matter when all operations are multiplication), so you can combine those two terms. But $xy^2$ and $x^2y^2$ are not like terms, so you cannot combine those without further information. Therefore, the simplest form here is $2xy + x^2y^2$.

5. What is $\frac{x(x + y) + 2(x + y)}{2 + x}$?
   Because $(x+y)$ is a common term in the numerator, you can factor it out: $(x + y)(2+x)$. Then the $(2+x)$ terms in the numerator and denominator cancel, leaving just $(x + y)$.
   **NOTE:** If the numerator were instead $2a + ax$, you would have immediately seen that you could factor out the common $a$ term. By flipping the order and having a compound $(x+y)$ term, the author of this question is betting that the more–complex setup will throw you off the scent. When you see common terms, no matter how complex, look to see if you can factor them. The more complicated the term, the nicer it is when you factor it out and don’t have to worry about it.
6. Simplify \( \sqrt{x} + \sqrt{4x} \).

We will more specifically cover roots later in the lesson, but at its core this is a factoring problem. \( \sqrt{4x} \) is the same thing as \( \sqrt{4} \cdot \sqrt{x} \), which means that the \( \sqrt{x} \) term is common and can be factored. Then you have \( \sqrt{x} \cdot (1 + \sqrt{4}) \).

Since you know that \( \sqrt{4} \) is 2, then that’s \( 3\sqrt{x} \).

7. Simplify \( 2 - (x + 4) + 2x - 4 \).

Again, here, be careful when distributing the negative sign across parentheses. This problem is essentially: \( 2 - 1(x + 4) + 2x - 4 \). That –1 is a helpful reminder that you’re multiplying all terms within the parentheses by -1. With that, you have:

\[
2 - x - 4 + 2x - 4
\]

Combine like terms: \( 2 - 4 - 4 - x + 2x = -6 + x = x - 6 \).

8. What is \( \frac{3 + 9x}{\frac{1}{3} + x} \)?

When you’re given an expression and not an equation, your only two options are to multiply by 1 or to combine like terms and factor. Here you can do either to simplify. You could multiply the top and bottom by 3 to get rid of the inner denominator (\( \frac{1}{3} \) on the bottom). That would give you \( \frac{9 + 27x}{\frac{1}{3} + 3x} \).

Then factor out that common 9 on the top to get: \( \frac{9(1 + 3x)}{\frac{1}{3} + 3x} \). The (1 + 3x) terms cancel, leaving just 9.

To do this just one step shorter, you can recognize in the original that the x in the denominator is really just the same thing as \( 3(\frac{1}{3})(x) \). That allows you to factor out the “common” \( \frac{1}{3} \) in the denominator, giving you: \( \frac{3(1 + 3x)}{\frac{1}{3}(1 + 3x)} \). Again, the common (1+3x) terms divide out, and the answer is 9.

9. If \( x^3 + x^3 + x^3 + x^3 = 40 \), what is \( x \)?

If you combine like terms, you’ll see that you have five \( x^3 \) terms, so \( 5(x^3) = 40 \). Divide both sides by 5 and you’ll find that \( x^3 = 8 \), meaning that \( x = 2 \). Note that what looks like a messy, abstract problem on first glance is actually quite quick when you combine like terms.

10. Simplify \( \frac{x + \frac{x}{2} + \frac{x}{3} + \frac{x}{8}}{15} \).

Here, in order to combine like terms, you first have to figure out the individual denominators on the top of the overall fraction. To eliminate them all at once, multiply the overall fraction by \( \frac{8 \cdot 8(x + 4x + 2x + x)}{15 \cdot 15} \). Combine like terms on top to find: \( \frac{8}{15} \), and you’ll see that the 15s cancel, leaving just \( \frac{x}{8} \).
11. Simplify \( \frac{7x + 14y + 28z}{28x + 14y + 7z} \).

At first glance, this may seem like it can get a lot simpler, but actually the best you can do is to factor out 7 in both the numerator and denominator to get: 
\[ \frac{7(x + 2y + 4z)}{7(4x + 2y + z)} \]. The 7s divide out, leaving \( \frac{x + 2y + 4z}{4x + 2y + z} \). Because there are no common terms within all portions of the numerator or denominator, you cannot reduce this any further.

12. Simplify \( \frac{58x - (17 + 7x)}{17} \).

First, distribute the multiplication across parentheses: 
\[ \frac{58x - (17 + 7x)}{17} = \frac{51x - 17}{17} \].

The common 17 on top and bottom should tip you off that 51x is likely divisible by 17, and it is: \( (51 = 3 \cdot 17) \). Thus, you can factor 17 out of the numerator: 
\[ \frac{17(3x - 1)}{17} \]. The 17’s divide out, leaving \( 3x - 1 \).

13. If \( x + x + x + x = x - 4x \), what is \( x \)?

Combine like terms to find that \( 4x = -3x \), and then that \( 7x = 0 \). Therefore, \( x \) must be 0.

14. What is \( \frac{x - 7}{7 - x} \)?

Here you should know that your best hope of eliminating the denominator is to factor. You can do so by factoring out \(-1\) from both terms in the denominator, making \( (7 - x) \) equal to \((-1)(-7 + x) \). Since \( (-7 + x) \) is the same as \( (x - 7) \), you now have common terms on both top and bottom: 
\[ \frac{x - 7}{-(x - 7)} \]. Those terms will then divide out, leaving \( \frac{1}{-1} \), which equals \(-1\).

15. When the sum of the first 20 positive multiples of 10 is divided by the sum of the first 20 positive even numbers, what is the result?

If you write out the first few terms of each, you’ll find that this problem looks like: 
\[ \frac{10 + 20 + 30 + 40 + \ldots + 200}{2 + 4 + 6 + 8 + \ldots + 40} \]. Because you know that division is easier when the numerator and denominator are multiplied instead of added, you should see an opportunity to factor. Since every term up top is a multiple of 10, there’s a common 10 that can be factored, and since every term on the bottom is a multiple of 2, there’s a common 2 that can be factored. So, really, this problem is: 
\[ \frac{10(1 + 2 + 3 + 4 + \ldots + 19 + 20)}{2(1 + 2 + 3 + 4 + \ldots + 19 + 20)} \]. The parenthetical terms are exactly the same, so they will cancel, and you’re left with \( \frac{10}{2} = 5 \).
Do the Same to Both Sides

To this point, you’ve covered mainly the skills for manipulating expressions, not for whole equations. When you’re only given an expression with no equal sign (like “Simplify $3(x + 2)$”) you can only combine like terms, factor, and multiply by 1. But when you’re given an equation, you have another important tool in your toolbox. As long as you do the exact same thing to both sides of the equation you will preserve that truth.

For example, note that everyone agrees that $6 = 6$. So if you have that equation, as long as you do the exact same things to both sides you’ll preserve that true statement:

$$6 = 6$$

- Multiply both sides by 2: $12 = 12$
- Add 4 to both sides: $16 = 16$
- Divide both sides by 2: $8 = 8$
- Square both sides: $64 = 64$
- Divide both sides by 4: $16 = 16$
- Take the square root of both sides: $4 = 4$

While that may seem obvious, with algebra you’re doing the exact same thing. You’re just doing so strategically to get rid of a variable. So when an equation is given such as:

$$\frac{x^2 - 4}{2} = 16$$

You’ll use the same logic to solve for $x$.

- Multiply both sides by 2: $x^2 - 4 = 32$
- Add 4 to both sides: $x^2 = 36$
- Solve the quadratic: $x^2 - 36 = 0$
  $$(x + 6)(x - 6) = 0$$
  $$x = 6 \text{ or } -6$$

The important things to remember when manipulating by doing the same thing to both sides are:

1. Be strategic: Your goal is to either isolate a variable or to make your algebra look like that in the answer choices.
2. Do the same thing to both entire sides, not to any one term while omitting another.
Consider this example:

\[
\frac{x - 4}{3} = 2(x + 2) - 22
\]

To solve for \(x\), you need to eliminate both the denominator and the parentheses. Be careful: In order to eliminate the denominator (usually a good first step), you must multiply the entire right side of the equation by 3—not just that first \(2(x + 2)\) term! It’s often helpful to put the entire side in parentheses to ensure that you distribute the multiplication accordingly:

\[
x - 4 = 3(2(x + 2) - 22)
\]

Conventional wisdom says to eliminate parentheses from the innermost set first, so distribute that multiplication:

\[
x - 4 = 3(2x + 4 - 22)
\]

\[
x - 4 = 3(2x - 18)
\]

Then distribute the second set:

\[
x - 4 = 6x - 54
\]

Then add 54 to both sides and subtract \(x\) from each side:

\[
50 = 5x \quad \text{and} \quad x = 10
\]

Consider also this example:

If \(\frac{2x}{y} = z\), then what is the value of \(\frac{yz}{x}\)?

Often on the GMAT, which contains many abstract questions, your goal is to take algebraic statements and make them look like the algebra in the questions. Here you do the same thing to both sides—not to solve for one particular variable, but to make the statement look like the term in the question stem.

Multiply both sides by \(y\):

\[
2x = yz
\]

Then divide both sides by \(x\) to make that term look just like the question: \(2 = \frac{yz}{x}\)

**Note:** Manipulating algebra to solve for a combination of terms is a key skill on the GMAT, particularly with Data Sufficiency. The drill that follows will give you practice with this skill.
**Cross-Multiplication**

When you are presented with two, single fractions set equal to each other, you can shortcut a step by cross-multiplying the fractions—essentially multiplying both sides by each denominator. Consider this example:

\[
\frac{x + 5}{4} = \frac{x + 1}{3}
\]

To eliminate denominators, you could multiply both sides by 4, eliminating the denominator on the left:

\[x + 5 = 4\left(\frac{x + 1}{3}\right)\]. Then you could multiply both sides by 3, eliminating the denominator on the right:

\[3(x + 5) = 4(x + 1)\].

Because you ultimately multiply each side by both denominators, and because multiplying each side by its own denominator has the effect simply of eliminating that denominator from that side, leaving only the numerator, you can streamline the process by cross-multiplying each numerator by the opposing denominator. Here, you'd take \(\frac{x + 5}{4} = \frac{x + 1}{3}\) and multiply the left by 3 and the right by 4, leaving:

\[3(x + 5) = 4(x + 1)\]

Then you can solve as normal:

\[3x + 15 = 4x + 4\]

\[11 = x\]
Let's try one more example, beginning with:

\[
\frac{3}{2 + x} = \frac{4}{3 + x}
\]

If you simply multiply the left side by the right denominator \((3 + x)\) and the right side by the left denominator \((2 + x)\), you get:

\[3(3 + x) = 4(2 + x)\]

You can then solve as usual:

\[
3 \cdot 3 + 3x = 4 \cdot 2 + 4x
\]

\[
9 + 3x = 8 + 4x
\]

\[
1 = x
\]

**NOTE:** You can use cross–multiplication when—and only when—each side of the equation is a single fraction (or whole quantity, which you can treat as a fraction with a denominator of 1). If there are multiple terms on either side of the equal sign, you must first combine them so that each side is a single fraction before you cross-multiply.
Do the Same to Both Sides Drills

Solve for x in the following equations. Note that, for the purposes of this drill, no denominator will equal 0:

1. $5x - 2 = 2x + 13$

2. $\frac{3}{x} = \frac{9}{x+4}$

3. $4 - 12x = 8x - 26$

4. $5x - 15 = 3(10 + x) - 5$

5. $2(3x - 7) + 4x = 8(4 + x) - 20$

6. $\frac{8}{x} = \frac{12}{3x + 9}$

7. $\frac{3x}{4} - 1 = x - 3$

8. $\frac{2x}{3} + 2 = \frac{5x}{2} - 9$

In the following examples, take what is given and leverage that to answer the question:

9. If $x = 2$, $y = z + x$, and $x = 2z$, what is $\frac{y - z}{z}$?

10. If $\frac{2a + b}{a} = 1$, what is $a + b$?

11. If $\frac{a}{b} = c$, what is $\frac{bc}{2a}$?

12. If $yz = xz$, but $az \neq bz$, what is $x - y$?

13. If $\frac{b - c}{b} = 5$, and $a + c = b$, what is $\frac{a}{b}$?

14. If $x + y = y - x$, what is $\frac{x}{y}$?
Solutions to Do the Same to Both Sides Drills

1. \(5x - 2 = 2x + 13\).
   Add 2 to both sides: \(5x = 2x + 15\)
   Subtract 2x from both sides: \(3x = 15\)
   Divide both sides by 3: \(x = 5\)

2. \(\frac{3}{x} = \frac{9}{x + 4}\)
   Cross-multiply: \(3(x + 4) = 9x\)
   Distribute multiplication: \(3x + 12 = 9x\)
   Subtract 3x from both sides: \(12 = 6x\)
   Divide both sides by 6: \(2 = x\)

3. \(4 - 12x = 8x - 26\)
   Add 26 to both sides: \(30 - 12x = 8x\)
   Add 12x to both sides: \(30 = 20x\)
   Divide both sides by 20: \(\frac{3}{2} = x\)

4. \(5x - 15 = 3(10 + x) - 5\)
   Distribute multiplication: \(5x - 15 = 30 + 3x - 5\)
   Combine like terms: \(5x - 15 = 25 + 3x\)
   Add 15 and subtract 3x from both sides: \(2x = 40\)
   Divide both sides by 2: \(x = 20\)

5. \(2(3x - 7) + 4x = 8(4 + x) - 20\)
   Distribute multiplication: \(6x - 14 + 4x = 32 + 8x\)
   Combine like terms: \(10x - 14 = 12 + 8x\)
   Subtract 8x and add 14 to both sides: \(2x = 26\)
   Divide both sides by 2: \(x = 13\)

6. \(\frac{8}{x} = \frac{12}{3x + 9}\)
   Cross-multiply: \(8(3x + 9) = 12x\)
   Distribute multiplication: \(24x + 72 = 12x\)
   Subtract 24x from both sides: \(72 = -12x\)
   Divide both sides by -12: \(-6 = x\)

7. \(\frac{3x}{4} - 1 = x - 3\)
   Add 1 to both sides to get the fraction on its own: \(\frac{3x}{4} = x - 2\)
   Multiply both sides by 4 to remove the denominator: \(3x = 4(x - 2)\)
   Distribute multiplication: \(3x = 4x - 8\)
   Subtract 4x from both sides: \(-x = -8\)
Divide both sides by -1:

\[
\frac{2x}{3} + 2 = \frac{5x}{2} - 9
\]

Add 9 to both sides:

\[
\frac{2x}{3} + 11 = \frac{5x}{2}
\]

Multiply both sides by 6 to eliminate denominators:

\[
4x + 66 = 15x
\]

Subtract 4x from both sides:

\[
66 = 11x
\]

Divide both sides by 11:

\[
x = 6
\]

9. What is \(\frac{y-z}{z}\)?

Given information:

\[y = z + x, \text{ and } x = 2z\]

Your goal should be to eliminate the compound term \(y - z\). So manipulate \(y = z + x\) by subtracting \(z\) from both sides to see that \(y - z = x\).

So the question is really asking:

What is \(\frac{x}{z}\)?

We are told that \(x = 2z\), so we can plug in \(2z\) for \(x\):

What is \(\frac{2z}{z}\)?

The \(z\)'s divide out, so the answer is 2.

10. What is \(a + b\)?

Given information:

\[
\frac{2a + b}{a} = 1
\]

Your goal is to use the given information in the question, so first eliminate the denominator in the given information by multiplying both sides by \(a\):

\[
2a + b = a
\]

Now subtract \(a\) from both sides:

\[
a + b = 0
\]

The answer is 0.

11. What is \(\frac{bc}{2a}\)?

Given information:

\[
\frac{a}{b} = c
\]

Multiply both sides by \(b\) in the given information so you get the value for \(bc\), which you can then substitute into the question to get:

\[
a = bc \quad \text{What is } \frac{a}{2a}?
\]

That makes the question:

\[
\frac{a}{2a}
\]

The \(a\)'s cancel, making the answer \(\frac{1}{2}\).
12. What is \(x - y\)?
   Given information: \(yz = xz,\) but \(az \neq bz\)
   This question involves more conceptual algebra.
   If \(yz = xz,\) there are two possibilities:
   
   Either \(y = x\) (and in each case you're multiplying \(z\) by the same thing) or \(z = 0\) (and it doesn't matter what \(y\) and \(x\) are, because multiplying anything by 0 gives you 0). The second statement—that \(az\) is not equal to \(bz\)—proves that \(z\) is not 0, so we know that \(x = y,\) and so the answer to the question is 0.

13. What is \(\frac{a}{b}\)?
   Given information: \(\frac{b - c}{b} = 5,\) and \(a + c = b\)
   The given information tells you that \(\frac{b - c}{b} = 5,\) which gets you halfway to simplifying the question, as the denominator (\(b\)) is already in place. Therefore, your goal should be to take the second part of the statement and see if you can re-express the numerator, \(b - c.\) If you subtract \(c\) from both sides, then you have \(a = b - c.\) This means that you can substitute \(a\) for \(b - c\) in the first equation, giving you: \(\frac{a}{b} = 5.\)

14. What is \(\frac{x}{y}\)?
   Given information: \(x + y = y - x\)
   Here, as no numbers are given, your goal should be to express \(x\) in terms of \(y.\) To align the variables, add \(x\) to both sides and subtract \(y\) from both sides to get \(2x = 0.\) This tells you that \(x = 0,\) which means that \(y\) is irrelevant: 0 divided by anything is 0, so the answer is 0.
Eliminate Variables

When problems involve multiple variables, you can often simplify them by using the equations to eliminate a variable and create a simpler equation. For example, if you were given the equations:

\[ x + y = 6 \quad \text{and} \quad x = 2y \]

You could use the second equation and plug in that definition of \( x \) into the first:

\[ (2y) + y = 6 \]

Now you just have one variable and one equation, so you can solve for \( y \):

\[ 3y = 6, \text{ so } y = 2 \]

Plug that value back into either of the original equations to solve for \( x \):

\[ x = 2(2) = 4 \]

The general rule is that, when solving a problem with \( N \) variables, you will need \( N \) unique, linear equations to solve for these variables. Be sure, however, that the equations are unique! Note that the following two equations are essentially the same (just different ways of phrasing the same fact), and as such they would not permit you to solve for either variable:

\[ x + 5 = y \quad \text{and} \quad 2x + 10 = 2y \]

Because the second equation can be created simply by multiplying both sides of the first by 2, it does not add new information. In order to solve for either \( x \) or \( y \) in the first equation, you would need a second unique equation.

There are two ways to eliminate variables: the substitution method and the elimination method.
Substitution Method

1. Write an equation that expresses one variable in terms of the other variables.
2. Substitute this new expression into another of the given equations.
3. Repeat until you get a numerical value for one variable, and then substitute this value into the equation you generated in step 1.

Example

\[ x + y = 7 \quad \text{and} \quad x - y = 1 \]  \hspace{1em} \text{(two equations and two unknowns: x and y)}

Write an equation that expresses one variable in terms of the other variable(s):

\[ x - y = 1 \Rightarrow x = 1 + y \]

Substitute this new expression into another of the given equations:

\[ x + y = 7 \Rightarrow (1 + y) + y = 7 \Rightarrow 2y + 1 = 7 \Rightarrow 2y = 7 - 1 \Rightarrow y = 3 \]

Now that you have gotten a numerical value for one variable, substitute this value for it into the equation you generated in step 1:

\[ x = 1 + y \Rightarrow x = 1 + 3 \Rightarrow x = 4 \]
Elimination Method

Add or subtract both sides of two or more equations to solve the equation directly. This is permissible because both sides are equal; you will be adding or subtracting the same amount to both sides.

Example:

\[ x + y = 7 \]
\[ x - y = 1 \]

Add both sides of the two equations together to yield a new equation:

\[
\begin{align*}
  x + y &= 7 \\
  + \quad x - y &= 1 \\
  \hline
  2x &= 8 \\
  \rightarrow \quad x &= 4
\end{align*}
\]

Substitute this known value for \( x \) into either equation to solve for \( y \):

\[ x + y = 7 \rightarrow 4 + y = 7 \rightarrow y = 3 \]

Eliminate Variables Drills

Solve for \( x \) and \( y \):

1. \( x + y = 10 \quad x - y = 6 \)
2. \( 3x - y = 8 \quad x = -5y \)
3. \( 3x + 2y = 9 \quad y = x + 2 \)
4. \( 3x + y = 8 \quad x - 2y = -2 \)
5. \( 2y - 3x = 12 \quad y + 2x = 10 \)
Solutions:

1. \( x + y = 10 \quad x - y = 6 \quad x = 8; y = 2 \)

Here the elimination method works best. If you add both equations, the \( y \) terms cancel and you’re left with \( 2x = 16 \). So \( x = 8 \), and you can plug into either equation to find that \( y = 2 \).

2. \( 3x - y = 8 \quad x = -5y \quad x = \frac{5}{2}; y = -\frac{1}{2} \)

Here the substitution method works best. Plug in \(-5y\) for \( x \) in the first equation to get:
\[-15y - y = 8\]
\[-16y = 8\]
\[y = \frac{1}{2}, \text{ and plug back in to the second equation to find that } x = \frac{5}{2}\]

3. \( 3x + 2y = 9 \quad y = x + 2 \quad x = 1; y = 3 \)

Here, again, the substitution method is already partially done for you. Plug in \( x + 2 \) for \( y \) in the first equation to get:
\[3x + 2(x + 2) = 9\]
\[3x + 2x + 4 = 9\]
\[5x + 4 = 9\]
\[5x = 5, \text{ so } x = 1 \text{ and } y = 3\]

4. \( 3x + y = 8 \quad x - 2y = -2 \quad x = 2; y = 2 \)

Here either method takes a step to set up. To use the elimination method, double the first equation to get:
\[6x + 2y = 16\]
\[x - 2y = -2\]
Then add the 2 to get:
\[7x = 14, \text{ so } x = 2 \text{ and } y = 2\]

5. \( 2y - 3x = 12 \quad y + 2x = 10 \quad x = \frac{8}{7}; y = \frac{54}{7}\)

Here the substitution method may be quicker to set up. Transform the second equation to: \( y = 10 - 2x \), and then plug that in to the first equation:
\[2(10 - 2x) - 3x = 12\]
\[20 - 4x - 3x = 12\]
\[20 - 7x = 12\]
\[7x = 8, \text{ so } x = \frac{8}{7} \text{ and } y = \frac{54}{7}\]
Multiple Variables and Special Cases

As you have seen, solving for multiple variables will most likely require you to have as many equations as variables. There are a few exceptions.

Definitions: If one of the variables is constrained by a definition such as “x is a positive integer” or “y is a prime number,” the pool of possible solutions may be narrowed to a point where you can solve with fewer equations.

Inequalities: Inequalities, such as $x < 5$, can also limit the pool of possibilities and allow you to solve for multiple variables with fewer solutions.

Example: If $x$, $y$, and $z$ are prime numbers such that $x < y < z$, and $x + y + z = 10$, what is the value of $y$?

Solution: Because the definitions limit $x$, $y$, and $z$ to prime numbers, you can determine that only one combination of prime numbers will add to 10: $2 + 3 + 5$. The next largest prime number is 7, and could not be combined with any two other prime numbers to produce a sum as low as 10. Accordingly, your options are limited to a point that you can determine that $x = 2$, $y = 3$, and $z = 5$, so the answer to this particular question is 3.

Outside of these exceptions, a good rule of thumb is that, when faced with more variables than equations, you should:

1. Search for more equations, or
2. Reduce the number of variables.

BEWARE: The GMAT writers know that examinees, under stress to act quickly, tend to miss key words that would allow them to find additional equations or eliminate or consolidate variables. Don’t!

Example

In preparation for a trip to New York, Xavier made two identical ATM withdrawals, and Yoni made one. Together, they withdrew exactly enough for each to contribute $60 to the travel fund. How much money did Xavier withdraw in his first transaction?

We know that $x_1 + x_2 + y = 120$, and that $x_1 + x_2 = 60$, but that still leaves us one equation short. The word "identical" alerts us that Xavier’s first withdrawal was the same as his second, so you can conclude that $x_1 = x_2$. The lesson? Words and phrases such as "identical" and the "same" are easy to overlook, but will help you to create new equations or consolidate variables in order to solve for multiple variables.
Exponents, Roots, Inequalities, and Quadratics

Your use of the Algebra Toolkit will help you with any algebra problem. However, other components of algebra require more content knowledge than just the Toolkit. In the pages that follow, you will gain a deeper familiarity with some of the other common applications of algebra on the GMAT.

Exponents

Exponent rules, and their application with variables, will be tested frequently on the GMAT. You should memorize them so that you have them mentally handy, but you should also understand what makes them work. The following table summarizes the rules you should know. With these and other rules, however, you should also note that rules must hold for all numbers (with occasional restrictions; in this case, for instance, these rules hold for all nonzero $x$ and $y$) and can therefore be tested with small numbers if you are unsure whether your memory is correct. As you refresh the rules on the next page, remember that understanding why a rule must hold true is the easiest way to ensure that you memorize it effectively.
<table>
<thead>
<tr>
<th>Rule</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x \cdot y)^a = x^a \cdot y^a)</td>
<td>The exponent distributes to all terms within the parentheses. To convince yourself, you can derive this rule quickly using smaller numbers ((4 \cdot 3)^2 = (4 \cdot 3) \cdot (4 \cdot 3)), and, since multiplication is both associative and commutative, that equals ((4 \cdot 4) \cdot (3 \cdot 3) = 4^2 \cdot 3^2).</td>
</tr>
<tr>
<td>(\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a})</td>
<td>This rule goes hand in hand with the above; again, the exponent distributes to all terms within the parentheses. A quick derivation: (\left(\frac{\frac{4}{3}}{\frac{4}{3}}\right)^2 = \frac{\frac{4}{3}}{\frac{4}{3}} \cdot \frac{\frac{4}{3}}{\frac{4}{3}} = \frac{4^2}{3^2}).</td>
</tr>
<tr>
<td>(x^a \cdot x^b = x^{a+b})</td>
<td>Try this with small numbers representing the exponents, such as (a = 3) and (b = 2): (x^3 \cdot x^2 = (x \cdot x \cdot x) \cdot (x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x = x^5).</td>
</tr>
<tr>
<td>(\frac{x^a}{x^b} = x^{a-b})</td>
<td>This is the counterpart to the above. Try using (a=5) and (b=2) and simplifying the fraction by cancelling: (\frac{x^5}{x^2} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x} = x \cdot x \cdot x = x^3).</td>
</tr>
<tr>
<td>((x^a)^b = x^{ab})</td>
<td>Again, convince yourself with small numbers. Try (a = 2) and (b = 3): ((x^2)^3 = (x \cdot x)^3 = (x \cdot x) \cdot (x \cdot x) \cdot (x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6 = x^{2 \cdot 3}).</td>
</tr>
<tr>
<td>(x^0 = 1)</td>
<td>Think of the fact that for the powers of a given number, each time you increase the exponent by 1, you are multiplying by the base again—so each time you decrease the exponent by 1, you are dividing through by the base. For instance, to get from (3^1) to (3^2), you multiply by 3; to get from (3^2) to (3^3), you multiply by 3 again; (3^3 = 3^1 \cdot 3), and so on. To step back down that list, therefore, you divide: To get from (3^4) to (3^3), you divide by three ((81 \div 3 = 27)); to get from (3^3) to (3^2), you divide by 3 ((27 \div 3 = 9)); to get from (3^2) to (3^1), you divide by three ((9 \div 3 = 3)). So to get from (3^1) to (3^0), you must divide by three again: (3 \div 3 = 1). Since any (x^1 = x) and, since to get from any (x^1) to (x^0), you must divide by (x), (x^0) will always equal (x \div x), which will always equal 1.</td>
</tr>
<tr>
<td>(x^{-a} = \frac{1}{x^a})</td>
<td>You know from the first rule that (x^a \cdot x^{-a} = x^{a-a} = x^0), and from the rule immediately prior to this one that (x^0 = 1). Putting these facts together, you get that (x^a \cdot x^{-a} = 1). We can then manipulate this equation by dividing by (x^a) on both sides to generate (x^{-a} = \frac{1}{x^a}). OR by dividing by (x^{-a}) on both sides to generate (x^a = \frac{1}{x^{-a}}).</td>
</tr>
<tr>
<td>(\frac{1}{x^a} = x^{-a})</td>
<td>This is the counterpart to the above.</td>
</tr>
</tbody>
</table>
Exponent Rules Drills

Use the rules above to manipulate the following expressions into equivalent expressions. You do not need to calculate final numerical values for the first seven problems—only write their equivalent expressions. (In other words, your answers may still contain exponents.)

1. \( 8^2 \cdot 8^3 = \)
2. \( (4 \cdot 7)^4 = \)
3. \( (5^3)^7 = \)
4. \( \left( \frac{2}{5} \right)^3 = \)
5. \( \frac{7^{11}}{7^5} = \)
6. \( 5^{-2} = \)
7. \( \frac{3}{7^{-1}} = \)
8. \( 11^0 = \)
9. \( 8^1 = \)
10. \( \frac{8^3}{4^5} = \)
11. \( 27^{29^{-3}} = \)
12. \( 2^{3^{1}} = \)
Solutions to Exponent Rules Drills

1. \(8^2 \cdot 8^3 = 8^5\)

2. \((4 \cdot 7)^4 = 4^4 \cdot 7^4\)

3. \((5^3)^7 = 5^{21}\)

4. \(\left(\frac{2}{5}\right)^3 = \frac{2^3}{5^3}\)

5. \(\frac{7^{11}}{7^5} = 7^6\)

6. \(5^{-2} = \frac{1}{5^2}\)

7. \(\frac{3}{7^{-3}} = 3 \cdot 7^3\)

8. \(11^0 = 1\)

9. \(8^1 = 8\)

10. \(\frac{8^3}{4^5} = \frac{(2^3)^3}{(2^2)^5} = \frac{2^9}{2^{10}} = 2^{-1} = \frac{1}{2}\)

11. \(27^{2 \cdot 9^{-3}} = (3^3)^2 \cdot 9^{-3} = 3^6 \cdot 3^{-6} = 3^0 = 1\)

12. \(2^{(3)^0} = 2^9\)

Remember order of operations: You must eliminate the parentheses first!
Exponent Drills

Simplify the following equations:

1. $4 \cdot x^3 \cdot 5x \cdot x^2y \cdot 5y$

2. $\frac{48z^4y^3}{12zy^2}$

3. $\frac{72x^3y^4z}{(2xyz)^3}$

4. $\frac{14x^2x^3y^4}{7x^4y^1}$

5. $(6x \cdot 2y^3)^2$

6. $\frac{2^3 \cdot 6^2}{18 \cdot 16^7}$

7. $\frac{15^3 \cdot 8}{2^2 \cdot 6^3}$
Solutions to Exponent Drills

1. \(4 \cdot x^3 \cdot 5x \cdot x^2y \cdot 5y = 100x^6y^2\)

2. \(\frac{48z^4y^3}{12zy^2} = \frac{4z^3}{y}\)

3. \(\frac{72x^3y^4z}{(2xyz)^3} = \frac{9y}{z^5}\)

4. \(\frac{14x^2x^3y^4}{7x^4y^{-1}} = 2x^3y^5\)

5. \((6x \cdot 2y^3)^2 = 144x^2y^6\)

6. \(\frac{2^3 \cdot 6^2}{18 \cdot 16^7} = 4\)

7. \(\frac{15^3 \cdot 8}{2^2 \cdot 6^3} = \frac{125}{4}\)
Roots

While exponents ask you to multiply a value by itself a number of times, roots do exactly the opposite. Roots, instead, ask: What number multiplied by itself (a certain number of times) will produce this value? The most common root, the square root, asks: What number, squared, will produce this value? Accordingly, the square root of 16 asks for a number that, when squared, will produce 16. That number is 4, as you see in the first example below. Even though all positive numbers technically have two square roots (a positive one and a negative one) the symbol $\sqrt{\cdot}$ represents only the “principal”—that is, the positive—square root. Furthermore, the word “principal” is often omitted, so that even when we say just “square root,” we are referring to the positive square root only.

Examples

If $x^2 = 16$, then $x = 4$ OR $-4$ because $4^2 = 16$ AND $(-4)^2 = 16$  BUT

$\sqrt{16} = 4$ only

$\sqrt{\frac{4}{25}} = \frac{2}{5}$ because $(\frac{2}{5})^2 = \frac{4}{25}$

$\sqrt{16 \cdot 36} = \sqrt{4^2 \cdot 6^2}$ because $(4 \cdot 6)^2 = 4^2 \cdot 6^2 = 16 \cdot 36$

$\sqrt{x^{12}} = x^6$ because $(x^6)^2 = x^{6 \cdot 2} = x^{12}$

$(\sqrt{x^6})^2 = x$ by definition. The process of finding the square root and then squaring it is analogous to adding a value to $x$ and then turning right back around and subtracting that value again, or to dividing $x$ by some value and then immediately multiplying the result back by that value. Just as addition and subtraction “undo” each other (i.e., cancel each other’s effect) and just as multiplication and division do as well, so, too, do taking square roots and squaring precisely counteract each other. Note here that only positive values of $x$ have real square roots, since it is impossible to multiply any real number by itself to yield a negative number (negative • negative = positive; positive • positive = positive).

$\sqrt{x^2} = |x|$, since regardless of whether $x$ itself is positive or negative, squaring it will yield a positive number, and then the $\sqrt{\cdot}$ will instruct us to take specifically the positive square root of that number.

**Note:** While there are many roots, which we will cover in the coming pages, the standard is the principal square root, and the radical sign, $\sqrt{\cdot}$, with no other notations, represents precisely that.
Roots Are Fractional Exponents

As you saw above, roots and exponents are closely linked, because roots represent a logical inverse of exponents. In fact, all roots can alternatively be represented as exponents, a convenient fact that will allow us to employ the exponent rules you've learned in the previous pages as methods to solve problems using roots.

For example, if you want to represent the square root of 16 exponentially—that is, if you want to represent $\sqrt{16}$ as $16^x$—then you need to figure out what value $x$ must have.

We know by definition that $\sqrt{(16)^2} = 16$. But you also know from our exponent rules that if $(16)^2=16$, that's the same as saying that $16^{2x} = 16^1$. Since these bases are equal, the exponent you are raising 16 to on the left side must be equivalent to the one you're raising it to on the right side, in order to preserve the truth of the equation. So it must be the case that $2x = 1$, and therefore that $x = \frac{1}{2}$. Thus, you can conclude that $\sqrt{16} = 16^{\frac{1}{2}}$.

By extension, let's suppose you want to represent $\sqrt[3]{5^3}$ exponentially (i.e., as $5^x$ for some value of $x$). You know that $(\sqrt[3]{5^3})^2 = 5^3$ (by definition). But you also know from our exponent rules that if $(5^3)^2 = 5^3$, then $5^{2x} = 5^3$, so it must be the case that $2x = 3$ and therefore that $x = \frac{3}{2}$. Thus, you conclude that $\sqrt[3]{5^3}$ is equivalent to $5^{\frac{3}{2}}$.

To summarize, there are three rules. Consider how the first rule is just a simpler-case version of the second, and the second is just a simpler-case version of the third:

1. $\sqrt{x} = x^{\frac{1}{2}}$
2. $\sqrt[3]{x} = x^{\frac{1}{3}}$
3. $\sqrt[3]{x^3} = x^{\frac{3}{3}}$

To move in the other direction (from a fractional exponent to a root), note that the numerator of the fractional exponent becomes the sole exponent, while the denominator of the fractional exponent indicates what root is to be taken. (To see this most clearly, read rule #3 above in reverse order, but also note that it applies to both rule #1 and rule #2, since the implied whole exponent is 1 in both cases, and since the root is the square (or “2nd”) root in the first case.)
Definitions for Factorizing Roots

Because all roots can be expressed as exponents, the rules that govern them are the same:

\[
\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}
\]

\[
\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}
\]

Examples

\[
\sqrt{\frac{1}{16}} = \frac{\sqrt{1}}{\sqrt{16}} = \frac{1}{4}
\]

\[
\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2}
\]

\[
\sqrt{\frac{1}{16}} = \frac{\sqrt{1}}{\sqrt{16}} = \frac{1}{4}
\]

\[
\sqrt{3} \cdot \sqrt{12} = \sqrt{3 \cdot 12} = \sqrt{36} = 6
\]

Addition and Subtraction

Add and subtract roots as you would any variable:

\[
x + x + x + x = 4x \quad \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} = 4\sqrt{2} \quad (4\sqrt{2} \text{ is the same as } 4 \cdot \sqrt{2})
\]

As with variables, you can only combine like roots:

\[
x + x + y = 2x + y \quad \sqrt{2} + \sqrt{2} + \sqrt{8} = 2\sqrt{2} + \sqrt{4 \cdot 2}
\]

But roots have one property that makes them unique when it comes to adding and subtracting: Because you can factor them, in many cases you know more about roots than you would about a variable. Remember that, based on the rules of factoring exponents:

\[
\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2}
\]

Since you can express the \(\sqrt{4}\) as 2, you can express \(\sqrt{8}\) as \(2\sqrt{2}\). Then the expression \(2\sqrt{2} + \sqrt{8}\) becomes \(2\sqrt{2} + 2\sqrt{2}\) and, since you now have like terms to combine, finally \(4\sqrt{2}\).
Roots Drills

1. Simplify $\sqrt{2} + \sqrt{8} + \sqrt{32}$

2. What is $\sqrt{\frac{4}{9}}$?

3. What is $\sqrt{16}$?

4. Simplify $\sqrt{27} + \sqrt{75} + \sqrt{48}$

5. What is $\sqrt{82}$?

6. What is $\frac{\sqrt{8}}{\sqrt{2}} + \sqrt{18}$?

7. If $\sqrt{x} + \sqrt{50} = \sqrt{16x} + \sqrt{8}$, what is $x$?

8. What is $\sqrt{\frac{36}{225}}$?

9. Simplify $\sqrt{343} + \sqrt{112} + \sqrt{63}$

10. What is $\frac{\sqrt{363} + \sqrt{300}}{\sqrt{75} + \sqrt{12}}$?
Solutions to Roots Drills

1. Simplify $\sqrt{2} + \sqrt{8} + \sqrt{32}$

   When simplifying roots, try to factor the root out into a square times something else. This can be expressed as:
   
   $\sqrt{2} + \sqrt{4 \cdot 2} + \sqrt{16 \cdot 2}$
   
   Since 4 and 16 are perfect squares, those will factor out:
   
   $\sqrt{2} + 2\sqrt{2} + 4\sqrt{2} = 7\sqrt{2}$

2. What is $\sqrt{\frac{4}{9}}$?

   This is the same as $\frac{\sqrt{4}}{\sqrt{9}}$, which is $\frac{2}{3}$.

3. What is $\sqrt{16}$?

   As you do with parentheses, work from the middle out. $\sqrt{16} = 4$, so this question is really asking for $\sqrt{4}$, which you know is 2.

4. Simplify $\sqrt{27} + \sqrt{75} + \sqrt{48}$

   When dealing with multiple roots added or subtracted, once you find one “base” that can’t be broken down further (here $\sqrt{27}$ is $3\sqrt{3}$; $\sqrt{3}$ cannot be broken down further), your goal is often to factor the other roots using that same base. That allows you here to break out: $\sqrt{9 \cdot 3} + \sqrt{25 \cdot 3} + \sqrt{16 \cdot 3}$. 9, 25, and 16 are all perfect squares, leaving you with $(3 + 5 + 4)\sqrt{3}$, or $12\sqrt{3}$.

5. What is $3\sqrt{8^2}$?

   Using the fact that roots are fractional exponents, you can see this problem as $(8^2)^{\frac{1}{2}}$, or $8^1$. As it’s likely easier for you to take the cube root of 8 than to take the cube root of 64, you can do that first (it’s 2) and then square that to get 4.

6. What is $\frac{\sqrt{8} + \sqrt{18}}{\sqrt{2} + \sqrt{32}}$?

   You should note that all terms are even, allowing you to factor each root into “something times 2” as a common term:
   
   $\frac{\sqrt{4 \cdot 2} + \sqrt{9 \cdot 2}}{\sqrt{2} + \sqrt{16 \cdot 2}} = \frac{2\sqrt{2} + 3\sqrt{2}}{\sqrt{2} + 4\sqrt{2}}$
   
   With all terms common, you can combine: $\frac{5\sqrt{2}}{5\sqrt{2}} = 1$. 
7. If \( \sqrt{x} + \sqrt{50} = \sqrt{16x} + \sqrt{8} \), what is \( x \)?

Here you should factor the numerical terms to isolate \( x \):

\[
\sqrt{x} + \sqrt{2 \cdot 25} = \sqrt{16 \cdot x} + \sqrt{2 \cdot 4} = \sqrt{x} + 5\sqrt{2} = 4\sqrt{x} + 2\sqrt{2}
\]

Combine like terms:

\( 3\sqrt{2} = 3\sqrt{x} \), so \( x = 2 \)

8. What is \( \sqrt{\frac{36}{225}} \)?

This is the same as \( \frac{\sqrt{36}}{\sqrt{225}} \), which is \( \frac{6}{15} = \frac{2}{5} \).

9. Simplify \( \sqrt{343} + \sqrt{112} + \sqrt{63} \).

With multiple roots it’s often easiest to start with the smallest one to find a common root to break down. You know that 63 = 7 • 9, with 9 as a perfect square, so you should look to divide the others by 7 to break those down:

\[
7\sqrt{7} + 4\sqrt{7} + 3\sqrt{7} = 14\sqrt{7}
\]

10. What is \( \sqrt{\frac{363 + 300}{75 + 12}} \)?

Again, look to break down roots into the same base.
You know that 300 = 3 • 100, so 3 is a good common start here.
This problem becomes:

\[
\frac{\sqrt{3 \cdot 121 + \sqrt{3 \cdot 100}}}{\sqrt{3 \cdot 25} + \sqrt{3 \cdot 4}} = \frac{11\sqrt{3} + 10\sqrt{3}}{5\sqrt{3} + 2\sqrt{3}} = \frac{21\sqrt{3}}{7\sqrt{3}} = 3
\]
Inequalities

Inequalities are similar to normal algebraic equations, except that the equal sign is substituted with one of the following symbols:

- $A \neq B$  \hspace{0.5cm} A is not equal to B.
- $A > B$ \hspace{0.5cm} A is greater than B.
- $A \geq B$ \hspace{0.5cm} A is greater than or equal to B.
- $A < B$ \hspace{0.5cm} A is less than B.
- $A \leq B$ \hspace{0.5cm} A is less than or equal to B.

All the rules that apply to equations also apply to inequalities (e.g., whatever operation you apply to one side of an equation you have to apply to the other side). The only difference with inequalities—and it’s an important difference—is that if you multiply or divide both sides by a negative number, you must flip the inequality sign.

For a demonstration of why the sign must be flipped, consider this:

"Two is greater than one." \Rightarrow 2 > 1

If you were to multiply both sides of the inequality by \(-1\), you would produce the expression:

\(-2 > -1\)

You know the above is incorrect; 2 is a larger value than 1, and so its opposite will be farther below 0 than will the opposite of 1. In order to account for this, you must flip the sign when multiplying or dividing by a negative:

\(-2 < -1\) \hspace{0.5cm} This is correct; \(-2\) is indeed less than \(-1\).

Examples

\[8 > 3 \Rightarrow (-1) \cdot 8 < (-1) \cdot 3 \Rightarrow -8 < -3\]

\[2x + 10 \geq 4 \Rightarrow 2x \geq 4 - 10 \Rightarrow 2x \geq -6 \Rightarrow x \geq -3\]

(Note that here you do not flip the sign because we are not dividing or multiplying by a negative number.)

\[3y - 3 < 5y + 7 \Rightarrow 3y - 5y < 7 + 3 \Rightarrow -2y < 10 \Rightarrow -\frac{2y}{2} > \frac{10}{2} \Rightarrow y > -5\]
Multiple Inequalities

When an expression features "bracketed" inequalities—three quantities separated by two inequality signs, as seen below—it is easiest to treat each inequality as a different statement for the purposes of performing algebraic operations:

\[ 2 < x + 2 < 4 \]

In the statement above, the algebra should be relatively straightforward. One would just need to subtract 2 from both sides to isolate \( x \). However with three quantities, there is no way to perform the same operation for "both" sides, and so you should break each inequality apart and treat this as two separate inequalities:

\[
\begin{align*}
2 < x + 2 & \quad \text{AND} \quad x + 2 < 4 \\
-2 - 2 & \quad \text{AND} \quad -2 - 2 \\
0 < x & \quad \text{AND} \quad x < 2 \\
\end{align*}
\]

Then, you can take these conclusions and reform the brackets: \( 0 < x < 2 \).

Here's another example:

If \( y - 2 < x < 2y + 1 \), what is the range of possible values for \( y \), in terms of \( x \)?

\[
\begin{align*}
y - 2 < x & \quad \text{AND} \quad x < 2y + 1 \\
+2 +2 & \quad \text{AND} \quad -1 -1 \\
y < x + 2 & \quad \text{AND} \quad x - 1 < 2y \\
& \quad \text{AND} \quad x - 1 < 2y
\end{align*}
\]

Again, you can reform the bracketing inequalities found above:

\[
\frac{x - 1}{2} < y < x + 2
\]
Inequalities Drills

Determine the value of x as specifically as possible by manipulating the inequality (or inequalities).

1. \( 12 + 3x > 15 + x \)

2. \( 0 > x \) and \( -\frac{1}{x} > 17 \)

3. \( 13 - 4x < 5 + 2x \)

4. \( 12 + y > 4 - x \) and \( 12 - 3y > 9 \)

5. \( 14 - x < y < 2x - 1 \)
Solutions to Inequalities Drills

1. \[ 12 + 3x > 15 + x \]
   \[ x > \frac{3}{2} \]

2. \[ 0 > x \text{ and } -\frac{1}{x} > 17 \]
   \[ -\frac{1}{17} < x < 0 \]

3. \[ 13 - 4x < 5 + 2x \]
   \[ x > \frac{4}{3} \]

4. \[ 12 + y > 4 - x \text{ and } 12 - 3y > 9 \]
   \[ x > -9 \]

5. \[ 14 - x < y < 2x - 1 \]
   \[ x > 14 - y; x > \frac{y + 1}{2} \]
Absolute Value

Absolute value, denoted by the symbol |n|, gives a quantity’s “distance from zero.”

On the number line, 3 and –3 are equally far from 0 on each side. Therefore, they have the same absolute value.

When using absolute value in algebraic terms, it is important to note that there are two possible “input” values behind every absolute value: if a variable has an absolute value of 5, for example, the variable could be either 5 or –5, because each of those values is five units away from zero:

|x| = 5 \(\rightarrow\) x could be 5 or –5, because |–5| = 5 and |5| = 5

Accordingly, any equation in which an expression within absolute value bars is equated to some numerical value needs actually to be seen as two possible equations with no absolute value bars: one that equates the quantity that was within the absolute value bars to the numerical value given, and another that equates the quantity that was within the absolute value bars to the opposite of the numerical value given.

\[
|\!x + 5\!| = 10 \rightarrow \begin{align*}
\text{x + 5} &= 10 \\
\frac{-5}{-5} &= 5 \\
\frac{x}{x} &= 5 \\
\text{OR} &\quad \frac{x + 5}{-5} = -10 \\
\frac{-5}{-5} &= -10 \\
\frac{x}{x} &= -10 \\
\end{align*}
\]

\[
|\!2x - 3\!| = 7 \rightarrow \begin{align*}
2x - 3 &= 7 \\
\frac{+3}{+3} &= 10 \\
\frac{2x}{2x} &= 5 \\
\text{OR} &\quad 2x - 3 = -7 \\
\frac{+3}{+3} &= -4 \\
\frac{2x}{2x} &= -4 \\
\end{align*}
\]
Absolute Value Drills

Solve for $x$.

1. $|15 - 3x| = 12$

2. $|5 - x| = 13$

3. $|x - 12| = 18$

4. $|3x + 2| = 11$

5. $|2x - 3| = 30 - x$
Solutions to Absolute Value Drills

1. \(|15 - 3x| = 12\)
   \[x = 1; x = 9\]

2. \(|5 - x| = 13\)
   \[x = -8; x = 18\]

3. \(|x - 12| = 18\)
   \[x = -6; x = 30\]

4. \(|3x + 2| = 11\)
   \[x = -\frac{13}{3}; x = 3\]

5. \(|2x - 3| = 30 - x\)
   \[x = -27; x = 11\]
Quadratics and Multiple Solutions

Earlier we covered problems with multiple variables. Other problems will include multiple solutions for the same variable. This is a common occurrence when an equation features a squared variable.

\(x^2 = 81\), for example, means that \(x\) is either 9 or \(-9\), as each value would satisfy the equation.

Why is this? Because any time you have a squared variable you should put the equation in the form \(ax^2 + bx + c = 0\), as shown below, and then solve for \(x\).

\(x^2 - 81 = 0\) and \((x + 9)(x - 9) = 0\) and \(x = 9\) or \(-9\)

Recognizing when you are dealing with quadratics and putting them in the form below is a key skill for the GMAT:

\(ax^2 + bx + c = 0\)

\(a\) and \(b\) represent coefficients. The best way to solve equations with multiple solutions is to express them as a series of terms multiplying to a product of 0. In this case, each individual term, when set equal to 0, represents a solution to the equation:

\((x + y)(x + z) = 0\)

As you may remember (fondly) from high school, you can multiply out expressions of the latter variety to eliminate the parentheses, using the FOIL method. FOIL is an easy way to remember to distribute the multiplication in such an expression across all terms:

First | Outside | Inside | Last
--- | --- | --- | ---
\(x \cdot x\) | \(x \cdot z\) | \(x \cdot y\) | \(y \cdot z\)

In this approach, you will multiply the first terms in each quantity, then the outside terms, then the inside terms, then the last terms, adding each product together.
Examples

(x + 4)(x − 2)
First

• x

Outside

• (−2)

Inside

• 4

Last

4 • (−2)

\[ x^2 − 2x + 4x − 8 = x^2 + 2x − 8 \]

(x + 7)(x + 2)
First

• x

Outside

• 2

Inside

• 7

Last

7 • 2

\[ x^2 + 9x + 14 \]
Factoring Quadratics

Because, when solving for quadratic equations, your goal will be to take the equations from the expanded format and express them instead as a product of terms equaling zero, you will want to employ the reverse of the FOIL method. Using the same principles, you will be able to take most quadratics on the GMAT and convert them to products.

Consider the example from the previous page:
\[ x^2 + 2x - 8 \]

If you want to convert it back to its “pre–FOIL” form, which will allow you to solve for its two solutions, you can use your knowledge of the FOIL method to do so:

The first term in the quadratic above will be created by the first terms multiplied, and the last terms will be created by the last terms, leaving the middle term to be formed by the outside and inside terms. Visually, you see:

<table>
<thead>
<tr>
<th>First</th>
<th>Outside/Inside</th>
<th>Last</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td>+2x</td>
<td>-8</td>
</tr>
</tbody>
</table>

Accordingly, the terms of our solution will need to be constructed to produce the above. In the expression below:

\[(x + a)(x + b)\]

The outside and inside terms need to create 2x, and the last terms need to create -8. Because the outside and inside terms each contain one x, you can simplify this search:

\[ ab \rightarrow \text{must equal } -8 \]

\[ a + b \rightarrow \text{must equal } 2 \]

Accordingly, your strategy should be to:

1. Determine the possible factors of the last term (ab): \( 1 \cdot 8, 2 \cdot 4 \)
2. Find a pair that adds to the outside/inside term (a + b): \( 4 + -2 \)
3. Enter these terms within the parentheses, and set each term equal to 0:

\[(x + 4)(x - 2) = 0\]

\[ x = -4 \text{ or } 2 \]

In summary, when factoring quadratic equations to the form \((x + a)(x + b)\), the a and b terms must multiply to the last number, and add to the middle coefficient:

\[ x^2 + (a + b)x + ab. \]
Factoring Quadratics Drills

Solve for x, y, or z by factoring the quadratic equation.

1. \( x^2 - 6x + 9 = 0 \)

2. \( x^2 - 9x + 14 = 0 \)

3. \( x^2 + 7x - 60 = 0 \)

4. \( z^2 + 18z + 77 = 0 \)

5. \( y^2 + y - 110 = 0 \)

6. \( y^2 - y - 56 = 0 \)

7. \( x^2 + 12x + 36 = 0 \)

8. \( y^2 - 11y - 42 = 0 \)

9. \( z^2 + 14z + 45 = 0 \)
Solutions to Factoring Quadratics Drills

1. \[ x^2 - 6x + 9 = 0 \]
   \[ (x - 3)(x - 3) = 0 \]
   \[ x = 3 \]

2. \[ x^2 - 9x + 14 = 0 \]
   \[ (x - 7)(x - 2) = 0 \]
   \[ x = 2; x = 7 \]

3. \[ x^2 + 7x - 60 = 0 \]
   \[ (x + 12)(x - 5) = 0 \]
   \[ x = -12; x = 5 \]

4. \[ z^2 + 18z + 77 = 0 \]
   \[ (z + 11)(z + 7) = 0 \]
   \[ z = -11; z = -7 \]

5. \[ y^2 + y - 110 = 0 \]
   \[ (y + 11)(y - 10) = 0 \]
   \[ y = -11; y = 10 \]

6. \[ y^2 - y - 56 = 0 \]
   \[ (y + 7)(y - 8) = 0 \]
   \[ y = -7; y = 8 \]

7. \[ x^2 + 12x + 36 = 0 \]
   \[ (x + 6)(x + 6) = 0 \]
   \[ x = -6 \]

8. \[ y^2 - 11y - 42 = 0 \]
   \[ (y - 14)(y + 3) = 0 \]
   \[ y = -3; y = 14 \]

9. \[ z^2 + 14z + 45 = 0 \]
   \[ (z + 9)(z + 5) = 0 \]
   \[ z = -9; z = -5 \]
LESSON

Introduction to Algebra

Of the core skills that lead to success on the GMAT, algebra tends to be the one that provides the highest return-on-study-investment. Students who are able to quickly recognize and manipulate common algebraic expressions and equations have a huge advantage over those who don’t, as so many GMAT Quantitative problems involve algebra. In the Arithmetic lesson, you learned that the GMAT does not place a huge reward on your ability to calculate numbers, but rather on your conceptual understanding of core arithmetic concepts. With algebra, you are rewarded for speed of recognition and for your facility with common algebraic expressions, equations, and skills. Accordingly, you should spend a good amount of time and effort getting yourself back to the level of algebraic proficiency that you had in high school.

Before you start this lesson, it bears repeating that the GMAT is not a proficiency test; it is a sorting mechanism for business schools to help determine which candidates are the most capable of success and which candidates are the most deserving of admission. However, the competitive standard for algebra on the GMAT is high, and this is a place where your content knowledge must be strong; students who hope to “fake it” with algebra and get by on shortcuts, guesswork, and back-solving are at a distinct disadvantage. The bottom line is that a high level of algebraic proficiency is a necessity on the quantitative section, so make sure you commit the time to meet that standard. In this lesson, you will review those essential skills and gain valuable insight as to how to use them.
Algebra and the Veritas Prep Pyramid

With algebra, the bottom and middle of the pyramid are very important. You must be proficient with algebraic manipulation in order to climb the pyramid. Given that, it is essential that you do many drills and problems to reinforce algebraic proficiency. The following skills and/or takeaways will be particularly highlighted in the different sections of this book:

“Core Skills” from Skillbuilder
- General algebraic manipulation of expressions and equations
- Creating equations in basic word problems
- Exponents and roots
- Factoring
- Quadratic equations
- Common algebraic equations
- Inequalities

“Skills Meet Strategy” Takeaways from the Lesson Section
- Guiding principles of algebra
- How to leverage algebra assets
- Picking numbers and isolating patterns
- Learning by Doing

“Think Like the Testmaker” Takeaways from the Lesson Section
- “An Inconvenient Truth”
- Abstraction
- Reverse-engineering
- Large/awkward numbers
- Exploiting common mistakes
When people struggle with algebra, it tends to be because they are trying to re-create algebra as a series of disconnected skills and rules. Algebra, one could argue, exists less as a series of man-made rules and more as the application of several important logical principles. Seeing logic within algebra will help you to understand, and not just memorize, the rules of the game. Recognize this about the GMAT and algebra: What the GMAT tests in algebra is generally "An Inconvenient Truth." In other words, the test will provide you with algebraic expressions or equations in an inconvenient form, requiring you to use your knowledge of algebra to make that statement more convenient. The following example highlights this concept of "An Inconvenient Truth":

1. \( \frac{5 + 5\sqrt{5}}{10 + \sqrt{500}} = \)

(A) \( \frac{1}{2} \)
(B) 2
(C) \( 1 + \sqrt{5} \)
(D) \( 1 + 5\sqrt{5} \)
(E) \( 5 + \sqrt{5} \)
LEARNING BY DOING
Manipulating Expressions

This complicated algebraic expression is really just $\frac{1}{2}!$ How do you know that? By applying rules of algebra that follow several important guiding principles (which you will learn shortly). First simplify the $\sqrt{500}$ into $10\sqrt{5}$ so that the expression looks like this: $\frac{5 + 5\sqrt{5}}{10 + 10\sqrt{5}}$. Then factor out the greatest common factor from the numerator and the denominator to create multiplication: $\frac{5(1 + \sqrt{5})}{10(1 + \sqrt{5})}$. Lastly, cancel the $1 + \sqrt{5}$ on the top and bottom to get the final answer of $\frac{1}{2}$. So much of algebra on the GMAT requires this type of manipulation and fluency with algebra. Answer choice A is correct.

THINK LIKE THE TESTMAKER
“An Inconvenient Truth”

The mechanics of algebra take practice. The overall goal, though, is something you can reason through. You were given an inconvenient piece of factual information, and your job was to clean it up and make it useful. Like a private equity firm restructuring a failing business or a consultant offering suggestions for an increased workflow, you will use algebra to restructure what you are given to put that factual information in a better position to solve a problem. That is why the GMAT so expressly tests algebra; in many ways, it’s a fitting metaphor for business.

SKILLBUILDER
• Simplifying roots
• Algebraic factoring
SECTION 1: THE ALGEBRA TOOLKIT

The GMAT can provide you with algebraic information in three basic ways: expressions, equations, and inequalities. It is important to note the form in which information is given to you. For example, in an expression without an equation, you don’t have as many tools at your disposal.

\[ a + \frac{2a}{3} \]

What is \( \frac{5}{3} \)?

This expression does not allow you to multiply both sides by \( 5/3 \) to eliminate the denominator. Why not? Because there is no other side to the equation. This is just a one-term expression, so your Toolkit here is limited. When you are only given an expression, and not an equation, you have two options: multiply by 1 or combine like terms/factor.

1. **Multiply by 1**

2. **Combine Like Terms/Factor**

Once you have an equation, a third option presents itself; then, you can use the equation to do the same thing to both sides of the equation, preserving the equality but rephrasing the statement: do the same thing to both sides.

3. **Do the Same to Both Sides**

When you have multiple variables, you can use the substitution or elimination method (more to come on those shortly) to reduce the number of variables.

4. **Eliminate Variables**

Some specific caveats will exist for inequalities, absolute values, and exponents, but in general these are the tools at your disposal. In the pages that follow, we will further explore these strategies and show you how to use them to better phrase algebraic expressions and equations to solve challenging problems.
Multiply by 1

Certain numbers have almost magical properties. You have likely already seen that the number 0 has unique properties that make it a major player on the GMAT. Another such number is 1, namely because, when you multiply another number by 1, you keep it the same. Strategically, this has quite a bit of value for you, as you can manipulate values algebraically by multiplying by the same numerator and denominator. For example:

What is \( \frac{a + 2a}{3} \)?

LEARNING BY DOING

Be Smart Multiplying by 1

Here again note that you don’t have another side to an equation. Your only options are to either combine like terms (the \( a \) terms in the numerator) or to multiply by 1 to help get rid of the denominator. As a matter of strategy, most of the time algebraic expressions look much, much simpler if you eliminate denominators first. But be careful. Another important axiom for GMAT algebra is this: keep it simple. Resist the temptation to do too much too soon. With this expression, it might seem wise to eliminate the entire denominator at once by multiplying by \( \frac{3}{5} \). But that’s a messy step. Much more cleanly, you can multiply top and bottom by 3:

\[
\frac{a + 2a}{3} \cdot \frac{3}{3} = \frac{3a + 2a}{5} = a
\]

NOTE: The first step (multiplying by \( \frac{3}{5} \)) is the same as multiplying by 1. It’s done strategically, to eliminate that messy denominator of \( \frac{3}{5} \). Also recognize this: The GMAT has a typical design to math problems in which the first two or three steps look ugly, but then problems drastically simplify after step 3 or 4—if you’ve taken the proper steps.

SKILLBUILDER

- General algebraic manipulation
How to Simplify Efficiently

\[ Y = \frac{4}{m} \div \left( \frac{1}{m} + \frac{2}{x} \right) \]

2. In the expression above, what is the value of \( Y \) if \( xm \neq 0 \) and \( m = \frac{1}{2} \cdot x \)?

(A) 1

(B) \( \frac{3}{2} \)

(C) \( \frac{5}{3} \)

(D) 2

(E) 4
LEARNING BY DOING
Manipulating Expressions

In this problem, simplify the right side of this equation by eliminating denominators. If you multiply the top and bottom by the lowest common multiple of the denominators (here, that is mx), that fairly intimidating expression is now: $\frac{4x}{x + 2m}$. By then substituting $\frac{1}{2}x$ for m, you see that the expression is really $\frac{4x}{2x}$ so $Y = 2$. Answer choice is D is correct.

While there are numerous ways you might manipulate this expression (combining the bottom terms first and multiplying by 1 in a different form), this is by far the cleanest and simplest approach. Algebra can be a bit of an art form; while there may be different ways to manipulate, one way is usually much more elegant.

SKILLS MEET STRATEGY
Get Rid of Fractions in Algebra

As a general rule, whenever you are faced with fractions in any algebra problem, get rid of those fractions with one step. Generally, that involves either multiplying by 1 (when you have an expression) or multiplying both sides of an equation (when you have a full equation) with the lowest common multiple (LCM) of the denominators. In this example, the LCM of x and m is mx, so you should multiply by 1 in the form $\frac{mx}{mx}$ to eliminate the first set of fractions. While fractions are your friends in arithmetic calculations, they are not in algebra!

SKILLBUILDER

• General algebraic manipulation
Combine Like Terms/Factor

As you’ve seen, the primary way that the authors of the GMAT make algebra-based questions difficult is by making the given information inconvenient. Often the most inconvenient structures are those with the most terms; much like you will find with Sentence Correction and Reading Comprehension, the more “things” you have to process, the more time a problem tends to take you and the more opportunities you have to make mistakes.

As a result, one of your goals with algebra questions is to combine like terms to streamline your operations. Combining like terms takes a few forms.

Most commonly, you can take any term that features the same variable to the same exponent and add those together:

\[ 4x + 3x = 7x \]

\[ 3y^2 - y^2 = 2y^2 \]

The following question tests both factoring and basic exponent rules. While we have not yet covered exponent rules specifically, this problem will refresh several core exponent rules from the Skillbuilder.

3. \[ 4^8 + 4^8 + 4^8 + 4^8 = 4^x \]. What is \( x \)?

(A) 4

(B) 8

(C) 9

(D) 16

(E) 32
LEARNING BY DOING

Combine Like Terms

In any algebra problem in which you have addition or subtraction, there are only two tools in your Toolkit:

1. You can combine like terms, as described in the section above. What is \( y + y + y + y \)? 4y. What is one chicken + one chicken + one chicken? Four chickens. Remember that you cannot combine unlike terms. What is \( y + y + y + z \)? 3y + z. What is one chicken + one chicken + one chicken + one dog? Three chickens and a dog.

2. You can factor out the greatest common factor (the largest thing in common with each term) from each term in the expression. What is \( 2^7 + 2^6 + 2^5 \)? \( 2^3(2^2 + 2^1 + 1) = 7(2^3) \)

In reality, factoring and combining like terms are the same concept. In the previous example (“What is \( y + y + y + y \)”) you could factor out the \( y \) from each term to get \( y(1 + 1 + 1 + 1) \) or 4y. Clearly that is an obtuse, but also correct, way to think about that question.

On this question, people tend to forget about combining like terms/factoring, and they make up their own exponent rules (to be covered shortly). To start the problem, you must simplify the left side of the equation: \( 4^8 + 4^8 + 4^8 + 4^8 = 4(4^8) \). Now that you have multiplication \( 4(4^8) \) you can then add exponents to see that expression equals \( 4^9 \). If \( 4^9 = 4^x \), then \( x = 9 \). Answer choice C is correct.

THINK LIKE THE TESTMAKER

Exploiting Common Mistakes

**NOTE:** \( 4^8 + 4^8 + 4^8 + 4^8 \) does not equal \( 4^{32} \), as some people will erroneously think. (You can only add exponents when you are multiplying terms.) Most algebra problems are set up to exploit one particular common algebra flaw. Don’t fall for it!

SKILLS MEET STRATEGY

When to Factor

On algebra problems with addition or subtraction of variables, exponents, and roots, you should only be thinking about factoring and/or combining like terms. By factoring and/or combining like terms, you will create multiplication and then be able to apply basic algebra rules.

---

**SKILLBUILDER**

- Exponent rules
- Factoring
More on Factoring

You will find that, particularly when dealing with roots and exponents, you may have to “create” like terms by factoring. Most notably, this is done by breaking complex numbers down into factors. For example, consider the following problem, which also introduces a few core skills with roots:

4. What is $\sqrt{363} + \sqrt{147} + \sqrt{27}$?
   (A) $6\sqrt{6}$
   (B) $7\sqrt{6}$
   (C) $11\sqrt{3}$
   (D) $21\sqrt{3}$
   (E) $30\sqrt{3}$
LEARNING BY DOING
Factoring with Roots

When problems require factoring, you should recognize clues that help you to factor more efficiently. With roots, as in this problem, you can factor the term $\sqrt{ab}$ into $\sqrt{a} \cdot \sqrt{b}$. In a vacuum, you can do this by recognizing common squares as factors of the given numbers ($363 = 3 \cdot 121$; $147 = 3 \cdot 49$; and $27 = 3 \cdot 9$), so they are simplified to: $11\sqrt{3} + 7\sqrt{3} + 3\sqrt{3}$ or $21\sqrt{3}$. Answer choice D is correct. With this problem, it is very important that you use the answer choices as a guide. It would be bad problem-solving strategy to stare at $363$ for two minutes trying to figure out how to factor that difficult number. By simplifying the easier one first ($\sqrt{27} = 3\sqrt{3}$) and looking at answer choices, it should be clear that the other two terms must leave behind $\sqrt{3}$, or there would be no way to combine the terms into the simple form of the answer choices. Remember: Algebra problems are not just about algebra; they are also exercises in problem solving that require smart approaches and different modes of thinking.

SKILLS MEET STRATEGY
Leveraging Assets

The main trick in this problem is that students will stare at $363$ for a long time trying to figure out how to factor that difficult number. On any GMAT problem, always start with the easiest piece of information to get the ball rolling. Here that means factoring $\sqrt{27}$ first, and leveraging that information to realize that $363$ can be broken down into $121$ and $3$. This strategy of leveraging easy information first is important in both Data Sufficiency (start with the easier statement) and multiple choice problems.

SKILLBUILDER
• Factorizing roots
• General factoring
Harder Factoring

Factoring is a difficult and important skill on the GMAT. Let’s look at a more challenging example that mixes algebraic factoring with some core arithmetic skills. Since we have not covered factorials yet, here’s a reminder of what a factorial is: \( n! = \text{the product of all positive integers up to and including } n \) (for instance: \( 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 \)).

5. What is the greatest prime factor of \( 12!1! + 11!10! \)?

(A) 2
(B) 7
(C) 11
(D) 19
(E) 23
LEARNING BY DOING

Turn Addition/Subtraction into Multiplication

To answer this question, you must turn addition into multiplication. For instance, if you were asked “What is the greatest prime factor of 11 + 17?” the answer does not lie in either of those two numbers individually. You must combine them to get 28 and then do the prime factorization to see that it is $2^2 \cdot 7$, and thus 7 is the largest prime factor. But you cannot simply combine these two expressions, as they are enormous numbers. Instead you must factor out from each expression. The general rule for factoring is this: Always factor out the greatest common factor within each term. On this problem, people tend to only factor out 11! from each term to get 11!(12! + 10!). That is okay, as long as you recognize that you can still factor out 10! from each of the remaining terms to get 11!10!(12 + 11 + 1). People who are adept at factoring would have done that as the first step, because 11!10! is the largest factor in common with each term. By factoring 11!10!, you have created multiplication, and you can simplify within the parentheses to get 11!10!(133). 133 is not prime (7 • 19), so that number must be broken down to create the simplified expression 11!10!(7)(19). Since the largest prime factor within 11!10! is 11, the largest prime factor of 11!10!(7)(19) must be 19. Answer choice D is correct.
Do the Same to Both Sides

To this point we have used mainly expressions, not equations, so we have only been able to multiply by 1 and factor/combine like terms. When equations are present, a third option emerges: You can do the same thing to both sides. This is permitted in algebra because of what the equal sign (=) means: The left side of the equation is the exact same value as the right side! So when you see something like this:

$$2x - 3 = 17$$

What you're really seeing is this:

$$17 = 17$$

The two sides are identical. The left side above is just much less convenient. Your job is to leverage that fact that $2x - 3$ is just another way to say "seventeen," using that equation to solve for the unknown $x$. As long as you do the exact same thing to the two exactly equal sides, you will not disturb that equality, and you can strategically perform operations to fill in the unknowns. So you can add 3 to both sides, then divide by 2, and find that $x = 10$.

You are allowed to do the same thing to both sides because you know that both sides of the equation are exactly the same. Beware the temptation to do something to just one term, and not to the entire side. For example, say you are given the following equation and asked to solve for $x$:

$$\frac{12(x - 2) - 7x}{21} = 4 - \frac{x}{3}$$
Note that many will accidentally multiply only the 4 on the right hand side by 21 when attempting to eliminate the denominator by multiplying both sides by 21.

Incorrect:

\[ 21\left(\frac{12x - 2}{21} - \frac{7x}{21}\right) = (21)4 - \frac{x}{3} \]

Correct:

\[ 21\left(\frac{12x - 2}{21} - \frac{7x}{21}\right) = 21(4 - \frac{x}{3}) \]

If you find yourself making this mistake, make a point to draw parentheses around either side of the equation before you begin multiplying/dividing both sides. When working hastily in a time-pressure situation like the GMAT, these quick-thinking mistakes can be score-killers, and the GMAT often has an incorrect answer choice teed up for those who fall victim to the most common errors.
Eliminate Variables

As you have learned, the GMAT likes to use abstraction and the appearance of large, “intimidating” numbers to make standard concepts appear more difficult and cumbersome. Accordingly, it will often employ multiple variables to make problems seem even more abstract. When multiple variables are present, you should keep this general rule in mind:

To solve for n number of variables, one needs n unique, linear equations.

**NOTE:** If you know something about the variables (that they are integers or in a certain range) it may be possible to solve for two unknowns with only one equation.

Multiple Ways to Solve for Multiple Variables

To be efficient with multi-variable equations, you should be adept with both the substitution method and the elimination method. Consider the example below:

**Substitution Method**

\[-3y = 5 - x\]
\[5x = 7y + 49\]

If one variable is not already isolated, structure one equation so that it isolates one variable expressed in terms of the other. For the equation \(3y = 5 - x\), isolate \(x\):

\[x = 3y + 5\]

Then, plug back in that rephrased value of \(x\) wherever \(x\) lies in the other equation:

\[5x = 7y + 49\]

Replace \(x\) with \((3y + 5)\):

\[5(3y + 5) = 7y + 49\]

\[15y + 25 = 7y + 49\]

\[8y = 24\]

\[y = 3\]

Then use either equation to plug in the \(y\) value and solve for \(x\):

\[x = 3y + 5\]
\[x = (3)(3) + 5\]
\[x = 14\]
Elimination Method

Often, equations will be offered within one or two steps of setting up an “elimination” for a variable. To employ elimination, you want to have two equations in which one equation adds the same variable that the other subtracts, such as:

\[ x + y = 7 \]
\[ x - y = 1 \]

Note that the top equation includes the term (+ y) and the other includes the term (-y). If you add the top equation to the bottom, the \( y \) terms will cancel:

\[ 2x = 8 \]
\[ x = 4 \]
\[ y = 3 \]

THINK LIKE THE TESTMAKER

Exploiting Common Mistakes

To employ either the substitution or elimination method, you must have multiple, unique equations. Testmakers know (particularly in data sufficiency) that if people see two equations with two unknowns, they will assume they can solve. But if an equation is simply a rephrasing of the other, it is not unique and therefore not helpful. For example, say you have:

\[ x = \frac{3}{5}y - 1 \]

and

\[ 5x = 3y - 5 \]

Those are the same exact equation, just rephrased. Given only one equation, you cannot solve for either \( x \) or \( y \).
Multiple Variables Drills

Having both methods at your disposal provides you with flexibility to find a quick way to solve. Consider the following examples and employ the method that is easiest:

1. \[3x + y = 12 - x\]
   \[6 = 2x - y\]

2. \[a = 2b - 1\]
   \[5a = 8b - 1\]

3. \[4m = 5n + 11\]
   \[13n = 21 - 2m\]
In the previous drill, #2 spots you the substitution without any work on your part while #1 hands you a in terms of b. But #3 is tricky to substitute. The coefficients for m and n are messy, but with a quick transformation of the first statement you can set up nicely for elimination:

\[ 4m = 5n + 1 \]

\[ 13n = 21 - 2m \]

By multiplying the entire second equation by 2, you can create an elimination in which the top equation has a + 4m and the other has a - 4m:

\[ 4m = 5n + 1 \]

\[ 42 - 4m = 26n \]

Combine the equations to find:

\[ 42 = 31n + 11 \]

\[ 31 = 31n \]

\[ n = 1, \text{ and } m \text{ must therefore be 4.} \]

Practice using both the substitution and elimination methods so that you can adeptly recognize and employ a strategy that will help you solve for variables quickly without much messy scratchwork.
Multiple Variables and Word Problems

Multi-variable equations often come in the form of word problems, in which you need to set up two equations. Consider this problem:

6. In a weightlifting competition, the sum of Draymond’s two lifts was 750 pounds. If twice the weight of his first lift was 300 pounds more than the weight of his second lift, what was the weight, in pounds, of his first lift?

   (A) 225
   (B) 275
   (C) 325
   (D) 350
   (E) 400
LEARNING BY DOING
Translating Wording and Solving Efficiently

First, use smart variables (f for first lift and s for second lift) to set up two equations:

\[ f + s = 750 \]
\[ 2f = s + 300 \]

Change the second equation to \[ 2f - s = 300 \] and use the elimination method:

\[ f + s = 750 \]
\[ 2f - s = 300 \]

\[ 3f = 1,050 \quad f = 350 \]

Answer choice D is correct.

SKILLS MEET STRATEGY
Use Smart Variables

In any word problem such as this, do not arbitrarily assign \( x \) and \( y \) to whatever you are solving for. Part of the trick in any problem like this is that people will accidentally solve for the wrong variable. To avoid that, always use a logical variable that matches what you are solving for in the word problem. In this example, for instance, use \( f \) for first and \( s \) for second.

SKILLBUILDER

- Translating wording into equations
- Eliminating variables
The Algebra Toolkit Summary

To succeed on GMAT algebra problems, it is essential that you are fluent in manipulating all algebraic expressions and equations. To do that, you must be intimately familiar with the Algebra Toolkit. The following summary covers all core skills and takeaways relating to this section:

**The Algebra Toolkit**

1. **Multiply by 1**
   
   Remember that multiplying by 1 tends to be a good first step whenever:
   
   - You’re provided with an expression, not an equation.
   - Your problem includes a multi-part denominator.
   - Your problem includes multiple terms and denominators.

   When you multiply by 1, you must be smart in choosing the form of 1 by which you multiply. Generally speaking, you want to choose the LCM of any terms in the denominators so that fractions disappear with one multiplication.

2. **Combine Like Terms/Factors**

   On algebra problems with addition or subtraction of variables, exponents, and roots, you should only be thinking about factoring and/or combining like terms. Remember when factoring that:
   
   - By factoring and/or combining like terms you will create multiplication and then be able to apply basic algebra rules.
   - You should always factor the greatest common factor (the largest thing in common with all terms).
   - In difficult and/or abstract factoring, consider simple examples to help you understand what to factor.

3. **Do the Same to Both Sides**

   Whenever you are dealing with an algebraic equation, you can do the same manipulation to both sides of the equation. You should consider the following when doing the same to both sides:
   
   - If fractions exist in the equation, multiply through by the LCM of the denominators to remove those fractions.
• Make sure that when multiplying or dividing, you properly manipulate each term within the equations.

• Be aware that careless errors are particularly common when manipulating both sides of an equation.

4. **Eliminate Variables**

   When multiple variables are present you must eliminate one or more variables to solve for one of them. To do that either:

   • Substitute. You should substitute if the equations are presented in a way that makes substitution the fastest and least error-prone approach.

   • Use the elimination method. You should use the elimination method when the equations can be easily lined up and combined to eliminate one or more of the multiple variables.
SECTION 2: EXPONENTS

As expressed in the Foundations of GMAT Logic lesson and in the beginning of this lesson, the authors of the GMAT have three main weapons to deploy to make fundamental math look exponentially more difficult:

1. Abstraction (the use of variables and symbols, and the presentation of concepts in less-concrete form)
2. Large/awkward numbers
3. Reverse-engineering (presenting a concept out of order from the way you’ve learned and practiced it)

One of the most common ways that the testmakers can employ tools 1 and 2 is to involve more exponents and more variables. Accordingly, it is to your advantage to become extremely comfortable with exponent-based algebra. Three “guiding principles” will lead you to success on any GMAT exponent-based problem:

1. Find common bases.
2. Multiply (which typically means “factor”).
3. Find patterns.
Exponent Rules Drills

Before we cover an example of each, let’s review the major exponent rules with this drill:

1. What is \((x)(x^4)(x^3)\)?

2. What is \((x^3)^5\)?

3. What is \(2^{3^2}\)?

4. What is \(2^3 \cdot 2^{-5}\)?

5. What is \(3^0\)?

6. What is \(\frac{4^2 \cdot 3^3}{12^2}\)?

7. If \(a\) and \(b\) are integers and \(3^{a^b} = 3^55^7\), can you solve for \(a\) and \(b\)?
Solutions to Exponent Rules Drill

1. $x^{2a+1}$ → When multiplying exponents of the same base, add their exponents together.

2. $x^{ab}$ → When taking one exponent to another, multiply the exponents together.

3. 512 (which is $2^9$) → With order of operations, one must perform parentheses functions first.

4. $\frac{1}{4}$ (which is $2^{-2}$) → A negative exponent $x^{-y}$ signifies to take the reciprocal $\frac{1}{x^y}$.

5. 1 → By definition, anything to the power of 0 is 1.

6. 3 → When dividing exponents (of the same base), subtract the exponents in the denominator from those in the numerator. Here, the denominator $12^2 = 4^2 \cdot 3^2$, so the 4 terms cancel and the subtraction leaves $3^1$ in the numerator.

7. Yes! Because 3 and 5 are prime numbers only that one exact combination can make it true; $a$ must be 5 and $b$ must be 7.
Find Common Bases

7. If $75^y27^{y+1} = 5^43^x$, what is the value of $x$?

(A) 8
(B) 10
(C) 12
(D) 15
(E) 17
LEARNING BY DOING
Exponent Rules and Finding Common Bases

If you review the common rules for dealing with exponents, you will find that most of them require a common base (or a single base) in order to combine like terms or otherwise rearrange the expression.

Accordingly, when you are presented with an exponent problem that features multiple bases, you should factor out each base to arrive at common bases, via which you can employ the rules above to make a more convenient statement. Most commonly, you will want to do this by reducing all bases to their primes. For this problem:

If $75y^27^{y+1} = 5^43^x$,

$75$ can be expressed as $3 \cdot 5^2$, and $27$ can be expressed as $3^3$, making the left side of the equation:

$(3 \cdot 5^2)^y \cdot (3^3)^{y+1}$

Distribute the parentheses: $3^y 5^{2y} \cdot 3^{3(2y+1)}$

Distribute parentheses once more: $3^y 5^{2y} \cdot 3^{6y+3}$

Combine like bases: $3^{7y+3} 5^{2y} = 5^4 3^x$

Your understanding of prime factors should help you recognize that it doesn’t matter how many 3s you multiply together—they’ll never create a 5. Thus, $5^{2y}$ must equal $5^4$, making $y = 2$. Accordingly:

$3^{7y+3} = 3^x$ then $3^{(2) + 3} = 3^x$ and $x = 17$ Answer choice E is correct.

SKILLS MEET STRATEGY
Find Common Bases on Exponent Problems

When you see multiple bases in an exponent problem, factor the bases to find common factors. And unless you quickly see common non-prime bases, your best bet is to factor each base down to primes.

SILLBUILDER
- Prime factorization
- Exponent rules
Multiply

8. What is $3^8 + 3^7 - 3^6 - 3^5$?

(A) $(3^5)(2^4)$

(B) $(3^5)(2^6)$

(C) $(3^6)(2^5)$

(D) $6^5$

(E) None of the above
LEARNING BY DOING
Factoring and Exponent Rules

Another thing that you should notice about exponent rules is that they all involve multiplication. After all, what is an exponent, really? It's a notation for a repetitive set of multiplication: $x^5 = \text{five } x\text{'s multiplied together: } x \cdot x \cdot x \cdot x \cdot x$.

When you see addition or subtraction in exponent problems, you should recognize that exponents are multiplication, and so you very likely will want to transform the expression into one that features multiplication. In this problem:

$3^8 + 3^7 - 3^6 - 3^5$ can be factored by recognizing the common $3^5$ within each term, and factoring that out to create a multiplied term:

$3^5(3^3 + 3^2 - 3 - 1)$

Now the numbers are in a more convenient form. You can quickly calculate each of the terms within the parentheses to find $27 + 9 - 3 - 1 = 32$. And how do you know to do that? For one, there isn't anything left to factor. You've already broken three of the terms down to single digits. Even more telling: look at the answer choices. Three of them feature a 2-to-an-exponent term, which at the very least should signify that you may want to turn that parenthetical into something even. Doing that, you should recognize that $32 = 2^5$, leaving $3^5(2^5)$—a value that is oh-so-close to answer choices A, B, and C. Where's the catch?

In the previous section, you learned to factor composite bases into primes. If you do that with $6^5$ in answer choice D, you will find:

$(2 \cdot 3)^5$

or $2^5 \cdot 3^5$, exactly what we have above. Using the same logic, you can take $3^5(2^5)$ and use the inverse of that property, recognizing that you essentially have $3 \cdot 3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. Combining those terms in a different order, you have five pairs of $2 \cdot 3$, which gives you $6^5$. Or you can just recognize the rule:

$x^y \cdot z^y = (xz)^y$ Answer choice D is correct.
SECTION 2: EXPONENTS

Multiply

SKILLS MEET STRATEGY
Focus on Core Competencies

You will find that a common theme with all GMAT questions is that they reward you for employing your core competencies—the handful of skills that you can perform extremely well—and for finding ways to use those core competencies even when they don’t present themselves obviously. That, again, harkens back to the theme of this lesson: “An Inconvenient Truth.” The GMAT rewards those who can make problems more convenient in order to apply those skills that they know how to do well. Your core competencies on exponent problems are the guiding principles: find common bases, turn addition/subtraction into multiplication/division, and, as you will see next, find patterns.

SKILLBUILDER

• Algebraic factoring
• Exponent rules
Find Patterns

As you saw in the previous problem, exponents are essentially just a notation for repetitive multiplication. And what happens when you do the same thing over and over again? You tend to get similar results. Exponents are particularly pattern-driven. When in doubt with an exponent-based problem, see if you can establish a pattern with small values, and then extrapolate it to the larger, given numbers. Consider this example:

9. What is the units digit of $2^{39}$?

(A) 2

(B) 4

(C) 6

(D) 8

(E) 9
Extrapolating Patterns

2^{39} is a massive number. You may have seen online or in Malcolm Gladwell’s book *The Tipping Point* that one need only fold a piece of paper in half 42 times to create enough height to reach the moon, taking advantage of the fact that exponential functions increase…well, exponentially. 2^{42} · the thickness of a sheet of paper equates to over 384,000 kilometers. So if your plan was to keep multiplying 2s (2, 4, 8, 16, 32, 64, 128, 256, 512…), you likely would have worn down less than halfway to calculating the full value of 2^{39}. That’s okay. The GMAT will never ask you to calculate such a massive value, but it will test whether you can predict what that number will look like. In fact, we’ve already done that in the short list above. We can establish a pattern with exponents of 2 and extrapolate it to 2^{39}. Look at the powers of 2 and the process to get there:

<table>
<thead>
<tr>
<th>2^1</th>
<th>2</th>
<th>Multiply by 2 to get:</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^2</td>
<td>4</td>
<td>Multiply by 2 to get:</td>
</tr>
<tr>
<td>2^3</td>
<td>8</td>
<td>Multiply by 2 to get:</td>
</tr>
<tr>
<td>2^4</td>
<td>16</td>
<td>Multiply by 2 to get:</td>
</tr>
<tr>
<td>2^5</td>
<td>32</td>
<td>Multiply by 2 to get:</td>
</tr>
<tr>
<td>2^6</td>
<td>64</td>
<td>Multiply by 2 to get:</td>
</tr>
<tr>
<td>2^7</td>
<td>128</td>
<td>Multiply by 2 to get:</td>
</tr>
<tr>
<td>2^8</td>
<td>256</td>
<td>And by now hopefully you’ve seen the pattern.</td>
</tr>
</tbody>
</table>

THINK LIKE THE TESTMAKER

Don’t Be Intimidated by Large or Awkward Numbers

What is the really big takeaway from this problem? Exponents are notoriously pattern-driven, so when you see large numbers or large exponents and need a way to proceed, see if you can test smaller versions of the same problem, establish a pattern, and then extrapolate it to fit the numbers that you have been given. Any time you are given large or awkward numbers on the GMAT, use a small, simple example to figure out what is going on. Using simple examples for bigger takeaways is one of the key strategies to success on the GMAT.
Because we're only concerned with the units digit, we've actually done some extra work since we hit 16. Just tracking the units digits, you can see that they follow a repeating pattern: 2, 4, 8, 6, 2, 4, 8, 6…. That will always recycle, because when you multiply 2 by 2 you get 4; when you multiply 4 by 2 you get 8; when you multiply 8 by 2 you get a 6 in the units digit (and carry the 1); and when you multiply 6 by 2 you get 2 in the units digit, and the cycle begins anew. Since we get four unique digits (2, 4, 8, 6, before coming back to 2 again), we know that the cycle repeats every fourth exponent. So 2 to the fourth, eighth, twelfth, sixteenth, etc. will end in a 6. Because 40 is a multiple of 4, then 2^{40} will end in a 6, and 2^{39} will be one before that in the cycle: It will end in an 8. The correct answer choice is D.

NOTE: Exponent units digit cycles have repeats of 1, 2, and 4. They are really just the times table so every time you get a problem, just write out the pattern for that particular number. For instance, the pattern for 4 is 4, 6, 4, 6… and the pattern for 8 is 8, 4, 2, 6, 8, 4, 2, 6…. 
Exponents Summary

When you face exponent-based problems, the three guiding principles for exponents, or some combination of them, will guide you to the answer. Keep these principles in mind:

1. Find common bases.
2. Multiply.
3. Find patterns.

Try the following problems as a capstone for our discussion of exponents, and explain:

• How they make these hard, and
• The value of making the abstract concrete with small numbers.

10. If $3^x - 3^{(x - 1)} = 2(3^{13})$, what is $x$?
    (A) 13  
    (B) 14  
    (C) 15  
    (D) 16  
    (E) 17

11. If $x = 5^{25}$ and $x^4 = 5^k$, what is $k$?
    (A) $5^{26}$  
    (B) $5^{27}$  
    (C) $5^{50}$  
    (D) $5^{52}$  
    (E) $5^{625}$
LEARNING BY DOING
Turn Subtraction into Multiplication or Find the Pattern

Given what you have learned so far, you should find one thing curious about problem #10: There is a 2 on the right side of the equation but no sign of a 2 on the left. Given the guiding principles of exponents, there must be some way to get common bases on each side. Since there is subtraction on the left side of the equation, you should immediately think about factoring so you can turn that side into multiplication (which should help you find the elusive 2). What makes this problem difficult is that the factoring is abstract and difficult. At this point you have several options:

A. If you are confused, try some numbers and see what happens: If x were 5, then the left side of the equation would be $3^5 - 3^4$. You would factor out $3^4$: $3^4(3 - 1) = (3^4)(2)$. If x were, say, 20, then the left side would be $3^{20} - 3^{19}$ and you would factor out $3^{19}$: $3^{19}(3 - 1) = (3^{19})(2)$. As you can see by trying a few numbers, if you factor out the $3^{x - 1}$ from the expression on the left, you will always be left with $(3^{x - 1})(2) = 2(3^{x - 1})$ and $x - 1 = 13$, so $x = 14$. Finding a pattern by using a few numbers makes the difficult abstract factoring much easier.

B. If you are more comfortable with abstract factoring, simply pull out $3^{x - 1}$ from each term on the left to see that you will get $3^{x - 1}(3^1 - 1)$, which is just $3^{x - 1}(2)$. You can then solve as above. This is quite difficult for most students to both see and execute.

C. If you do choose to factor abstractly, it is most likely you will see that there is a $3^x$ in common with each term on the left. This is because the expression on the left side of the equation can be written as $3^x - (3^x)(3^{-1})$. After pulling out $3^x$ you get $3^x(1 - 3^{-1})$, which is really $3^x(1 - \frac{1}{3})$, which equals $3^x(\frac{2}{3})$. Now look at the whole problem again rewritten as: $3^x(\frac{2}{3}) = 2(3^x)$. If you multiply both sides by 3 to eliminate the denominator you get $3^x(2) = 2(3^{14})$ and $x = 14$.

**NOTE:** The only problem with this approach is that by factoring out the larger number ($3^x$ instead of $3^{x - 1}$) you are left with a fraction, which makes a few more algebraic steps. The correct answer is answer choice B.

**SKILLBUILDER**
- Exponent rules
- Algebraic factoring
- Sequences → Algebra lesson
Roots Are Exponents, Too

In the factoring section, you were exposed to some basic root skills. It is important to remember that roots are just fractional exponents:

\[ \sqrt{x} = x^{\frac{1}{2}} \]

Accordingly, all of the properties of exponents—and the guiding principles that we’ve discussed—apply to roots.

### SUMMARY OF ROOT RULES

- When roots are divided: \[ \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \]
- When roots are multiplied: \[ \sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \]
- When roots are added/subtracted: \[ \sqrt{a} + \sqrt{a} = 2\sqrt{a} \]
- Roots are fractional exponents: \[ a^{\frac{1}{2}} = \frac{1}{\sqrt{a}} \]

Finding common “bases” and turning addition/subtraction into multiplication tend to be the most critical skills on root problems. Consider this tricky root problem that combines many skills we have learned thus far:

12. \[ \frac{\sqrt{180} + \sqrt{45}}{\sqrt{135}} = \]

   (A) \( \sqrt{3} \)
   (B) \( 3\sqrt{3} \)
   (C) \( 5\sqrt{3} \)
   (D) \( 3\sqrt{5} \)
   (E) \( 5\sqrt{5} \)
LEARNING BY DOING
Simplifying Roots and Rationalizing Denominators

There are two different approaches to this problem. In dealing with roots, the natural tendency is to first simplify the roots, and that will work here:

\[
\sqrt{180} = 6\sqrt{5}, \quad \sqrt{45} = 3\sqrt{5}, \quad \text{and} \quad \sqrt{135} = 3\sqrt{15},
\]

so the expression is now:

\[
6\sqrt{5} + 3\sqrt{5} = 9\sqrt{5} = 3\left(\frac{\sqrt{5}}{\sqrt{15}}\right) = 3\sqrt{\frac{5}{15}} = 3\sqrt{\frac{1}{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}.
\]

Answer choice A is correct.

It is a lot faster, though, to notice that there is a \(\sqrt{45}\) in each of the three expressions and then factor that out as follows:

\[
\frac{\sqrt{180} + \sqrt{45}}{\sqrt{135}} = \frac{\sqrt{45}(\sqrt{4} + 1)}{\sqrt{45}(\sqrt{3})} = \frac{3}{\sqrt{3}} = \sqrt{3}.
\]

THINK LIKE THE TESTMAKER
Misdirection

Another common ploy used by testmakers is “misdirection.” That is, they know how people will think or behave, and they exploit that. Since everyone learns to first simplify roots, the natural inclination on this problem is to take the first approach outlined above. However, it is clearly faster to use another known algebraic skill—general algebraic factoring—to simplify the expression more effectively. While it does not matter much which approach you use on this particular example, it will matter very much on other problems. Always take a little time to find the right approach before jumping into a problem, and remember: Your first inclination about how to attack a problem is almost always incorrect on hard problems!

SKILLBUILDER

- Factorizing roots
- Algebraic factoring
- Rationalizing denominator
Exponents Summary

For exponents and roots, it is essential that you understand all underlying rules. These are listed below and also can be found in the Skillbuilder.

Summary of Exponent Rules

When multiplying exponents with like bases: \(x^a \cdot x^b = x^{a+b}\)

When dividing exponents with like bases: \(x^a \div x^b = x^{a-b}\)

With parentheses: \((x^a)^b = x^{ab}\)

With multiplication/division and parentheses: \((x \cdot y)^a = x^a y^a\) and \((\frac{x}{y})^a = \frac{x^a}{y^a}\)

Summary of Root Rules

When roots are divided: \(\frac{\sqrt[a]{a}}{\sqrt[b]{b}} = \sqrt[ab]{\frac{a}{b}}\)

When roots are multiplied: \(\sqrt[ab]{a} \cdot \sqrt[bc]{b} = \sqrt[abc]{ab}\)

When roots are added/subtracted: \(\sqrt{a} + \sqrt{a} = 2\sqrt{a}\)

Roots are fractional exponents: \(a^{\frac{1}{a}} = \sqrt[a]{a}\)

Strategy for Exponent and Root Problems

While understanding rules is essential, it is determining the proper approach on these problems that makes them truly difficult. To succeed on exponent/root problems, follow these guiding principles:

- Find common bases on exponent problems.
- If a problem involves addition or subtraction, use factoring or combining like terms to create multiplication so that rules can be applied.
- On difficult exponent problems, find patterns and extrapolate. Exponent problems are inherently pattern-driven.
- On problems involving abstraction or large numbers, use simple examples or numbers in order to reason out the proper approach.
SECTION 3: QUADRATIC EQUATIONS

Quadratic equations blend the topics that we have already covered. They are equations that need to be made more convenient using the Algebra Toolkit, but they prominently feature exponents. You may remember the standard quadratic equation setup from your high school days:

\[ ax^2 + bx + c = 0 \]

**The Practical Approach: Factoring by Inspection**

The core skill on the GMAT for quadratics is the ability to take a quadratic in the form \( ax^2 + bx + c = 0 \) and factor that into a pair of multiplying factors. So given a quadratic like this one:

\[ x^2 - 6x - 27 = 0 \]

Factor the subtraction (or addition) to create a multiplication problem. Remembering FOIL from the Skillbuilder can be a hint: Your goals are to create parentheticals in the form:

\[(x + \_\_)(x + \_\_]\]

In which the “blanks”:

- Multiply to the last term (the c term in \( ax^2 + bx + c \)),
- Add to the coefficient of the middle term (b).

Accordingly, the process should be to:

1. Factor c, creating potential pairings of numbers.
2. Find the pairing that adds to term b, keeping in mind that you are responsible for the +/- sign, too.

Here, the factor pairings of 27 are (1, 27) and (3, 9). We need a term that adds to –6, and in which the terms have different signs to multiply to negative 27. –9 and 3 fit the bill, so the proper factorization is:

\[(x - 9)(x + 3) = 0\]
Be warned! By far the most common "silly mistake" when factoring quadratics is this: You might be tempted to answer: $x = -9$ or $x = 3$, but there's one more step! In order for the multiplication on the left to equal the 0 on the right, either $(x - 9) = 0$ or $(x + 3) = 0$. That means that either $x = 9$, or $x = -3$. People often leave this last step a step short. Be sure not to make this mistake!
Quadratic Factoring Drills

Before completing a full quadratic problem, consider the a few drill problems to reinforce this skill by solving for x.

1. \( x^2 + 13x + 36 = 0 \)
2. \( x^2 + 2x - 24 = 0 \)
3. \( x^2 = 9x \)

13. If 4 is one solution of the equation \( x^2 + 3x + k = 10 \), where \( k \) is a constant, what is the other solution?

(A) −7
(B) −4
(C) −3
(D) 1
(E) 6
**LEARNING BY DOING**  
*Factoring Quadratic Equations by Inspection*

The most common approach to this problem is to use the one solution (4) to solve for the constant. If \( x = 4 \), then \( k \) must be \(-18\). Plugging in \( k \), you see that the full quadratic is \( x^2 + 3x - 28 = 0 \). Factoring, you see that \((x + 7)(x - 4) = 0\), and the other solution is \(-7\). It is at this point that you might realize that you did not need to solve for \( k \). If 4 is one solution, then one side of the factored quadratic must be \((x - 4)\). Since the middle term is \(3x\), then the other factor must be \((x + 7)\), as that is the only way to get a positive 3x in the middle. The correct answer choice is A.

**THINK LIKE THE TESTMAKER**  
*Misdirection*

This problem is another good example of misdirection. Because there is an unknown constant, your natural inclination is to solve for it. However, with a good conceptual understanding of how quadratic formulas work, you do not need to even solve for it, because you know the middle term is \(3x\) and the other factor must be \((x + 7)\) to create that middle term. You are almost always rewarded with shortcuts on a GMAT problem if you understand a concept well, so take some time to look for that shortcut before jumping into a problem.
The Quadratic Formula: You Won’t Use it!

For quadratics in the form $ax^2 + bx + c = 0$, you can find the solutions employing the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

But recognize this: That formula is a long one to memorize if you don’t already have it committed to memory based on your prior math experience. It’s a bit tedious to employ. And the GMAT isn’t apt to require memorization for memorization’s sake. On the GMAT, two options should present themselves long before you ever have to employ the quadratic formula:

1. You can factor the equation using the method from the previous page.
2. You can plug in answer choices as potential values of $x$, probably much quicker than you could calculate using the quadratic formula.

Simply put, if you already have the quadratic formula memorized and in the “active” part of your brain, there’s a small chance that it could come in handy. If not, look for the opportunity to either factor or back-solve, and don’t worry about memorizing this formula.
Section 3: Quadratic Equations Summary

The skill set required for Quadratic Equations on the GMAT is relatively small. To succeed you must be able to:

- Recognize quadratics and put them in the form $ax^2 + bx + c = 0$. Remember that any equation with one squared variable is a quadratic and should be put in this form and factored.

- Factor by inspection any quadratic that can be broken down using this approach. To factor by inspection, find the pairs of factors that when multiplied together give you $c$ and when added together give you $b$.

- Back-solve when necessary. If you cannot factor by inspection, look at the answer choices and back-solve the problem. For almost all difficult quadratic equations on the GMAT, you can use this approach if you cannot factor by inspection.

- Understand how quadratics work but do not worry about memorizing quadratic equations on the GMAT.
SECTION 4: COMMON ALGEBRAIC EQUATIONS

As you saw in Section 3, equations with squared variables are much more complicated than simple linear equations. These structures are unique in that they represent the same parenthetical term squared, not one multiplied by another. There are two distinct common algebraic equations seen on the GMAT: perfect squares and the difference of squares.

Two Types of Perfect Squares

\[(a + b)^2 = a^2 + 2ab + b^2\]
\[(a − b)^2 = a^2 − 2ab + b^2\]

It is important to recognize that these are just templates for anything in the form \((a + b)^2\) or \((a − b)^2\). You always add the square of the first term, double the product of the two terms, and the square of the last term.

Common Algebraic Equations Drills

Show the equivalent form of the following expressions without FOILing:

1. What is \((2x + 5y)^2\)?
2. What is \((3y − 2x)^2\)?
3. What is \((\sqrt{3} + 1)^2\)?
4. What is \(25x^2 + 30xy + 9y^2\)

You should learn to recognize these structures without FOILing, as quick recognition can help you rapidly take an inconvenient expression and transform it into something useful.
14. If the expression \( x^2 - \frac{xy}{5} + 25 \) can be expressed by \((x - 5)^2\), what is the value of \( y \)?

(A) 0

(B) 5

(C) 25

(D) 50

(E) \( \frac{5}{x} \)
LEARNING BY DOING
Move Seamlessly Between Different Forms of the Common Algebraic Equations

To start this problem, you should recognize \((x - 5)^2\) as one of the common algebraic equations and know that transforming it into its FOILed form will be an important part of the problem. If \(x^2 - \frac{xy}{5} + 25 = (x - 5)^2\), then \(x^2 - \frac{xy}{5} + 25 = x^2 - 10x + 25\). By canceling the like terms on each side (25 and \(x^2\)), you see that \(10x = \frac{xy}{5}\) and \(50x = xy\) so \(y = 50\). Answer choice D is correct.

SKILLS MEET STRATEGY
When Common Algebraic Equations Are Given to You in One Form, Change Them into the Other Form

Almost without exception, when common algebraic equations are present on a GMAT problem in one form, the key to solving that problem will be to transform them into an equivalent form. Students who are able to quickly recognize and transform the common algebraic equations (particularly the difference of squares, as you will see shortly) have a huge competitive advantage on the test.

SKILLBUILDER
- Recognizing and transforming common algebraic equations
- Algebraic manipulation
Difference of Squares

The third of the common algebraic equations may be the most versatile and critical of the three for you to know for the GMAT. Often with difficult problems involving squared variables, the difference of squares acts as something of a “secret decoder ring,” allowing you to take a messy construct and turn it into something clean and manageable in just one step. To begin, consider a drill:

What is \((\sqrt{2} + 1)(\sqrt{2} - 1)(\sqrt{3} + 1)(\sqrt{3} - 1)\)?

While one could simply FOIL the pairs of parentheticals, employing the difference of squares rule makes this question run quickly. That rule, which you should commit to memory immediately, is:

\((x + y)(x - y) = x^2 - y^2\)

If you FOILed the problem above, you can see the genesis of difference of squares. Because the O and I terms are opposites, they cancel out, leaving just the F term minus the L term. But difference of squares is much more than simply a situational shortcut for FOIL. In many problems, it is plainly essential, and generally it is more important to recognize it “backward” (from quadratic to factored form). In the example above you can see that the answer is really just \((2 - 1)(3 - 1) = 2\).

With recognition only, show the equivalent forms of the following expressions:

1. What is \(16x^2 - 1\)?
2. What is \((5y + 2x)(5y - 2x)\)?
3. What is \(49y^2 - 9x^2\)?
4. What is \(\frac{25x^2 + 30xy + 9y^2}{25x^2 - 9y^2}\)?
More Difficult Difference of Squares

15. 999,999² – 1 =

   (A) 10¹⁰ – 2
   (B) (10⁶ – 2)²
   (C) 10⁵(10⁶ – 2)
   (D) 10⁶(10⁵ – 2)
   (E) 10⁶(10⁵ – 2)
LEARNING BY DOING

Look Out for the Difference of Squares

As with the other two common algebraic equations, any time that you see something in the form \( x^2 - y^2 \) on the GMAT, you must transform it. Note the difference of squares in the original expression: \( 999,999^2 - 1 = (999,999 + 1)(999,999 - 1) = (1,000,000)(999,998) \). Now you must use answer choices for the fairly arbitrary final step: \( 1,000,000 = 10^6 \) and \( 999,998 \) can be expressed as \( 10^6 - 2 \), so the correct answer choice is E.

THINK LIKE THE TESTMAKER

Large/Awkward Numbers

This problem is another good example of how testmakers perplex students with unmanageable numbers. When you are given incredibly large and awkward numbers such as \( 999,999^2 \), there is always some clean way to contend with that number. Often on the GMAT you deal with large numbers by considering a simple example. Here, though, you must recognize the difference of squares and then transform it into a more usable form.

SKILLBUILDER

- Recognizing and transforming common algebraic equations
- Algebraic manipulation
Common Algebraic Equations Summary

16. If \(a + b = x\), and \(a - b = y\), then \(ab = \)

(A) \(\frac{x^2 - y^2}{2}\)

(B) \(\frac{(x + y)(x - y)}{2}\)

(C) \(\frac{x^2 - y^2}{4}\)

(D) \(\frac{xy}{2}\)

(E) \(\frac{x^2 + y^2}{4}\)
LEARNING BY DOING
Be Clever with the Common Algebraic Equations

This problem requires clear recognition of the common algebraic equations. Because the problem is asking for the product ab, your “common algebraic equations” radar should be honed in on the middle term when you square either \((a + b)\) or \((a - b)\). If you square each one you see that:

\[(a + b)^2 = x^2 \quad \text{and} \quad (a - b)^2 = y^2\]

Rewritten these are: \(a^2 + 2ab + b^2 = x^2\) and \(a^2 - 2ab + b^2 = y^2\)

By subtracting the second from the first you see that:

\[a^2 + 2ab + b^2 = x^2 - (a^2 - 2ab + b^2 = y^2)\]

\[4ab = x^2 - y^2\]

Thus, \(ab = \frac{x^2 - y^2}{4}\). Answer choice C is correct.

SKILLS MEET STRATEGY
Leveraging Assets

This is another great example of how to apply knowledge on a GMAT problem. Simply memorizing the common algebraic equations is not enough on a problem such as this. You must leverage your knowledge that the term ab is an important component of the two perfect squares. By recognizing this, you then realize the need to square both \(a - b\) and \(a + b\) in order to create that middle term. How to apply knowledge, not the knowledge itself, is what makes the GMAT hard.
While this problem can be solved quickly by recognizing the common algebraic equations, it can also be solved by using an important problem-solving strategy: number picking. Assign smart, easy numbers to the variables $a$ and $b$ first: $a = 3$ and $b = 2$. Given those assigned values, $x = 5$, $y = 1$, and $ab$ (what you are solving for) must be 6. Go through each answer choice (you must check all of them, as multiple answer choices could appear to be correct) to find which one equals 6 with the assigned values for $a$, $b$, $x$, and $y$.

As you can see, there are two ways to solve this problem: on a conceptual basis, by understanding and recognizing common algebraic equations OR using good problem-solving technique, by “gaming” the problem with number picking. High scorers on the GMAT have one thing in common: They are flexible problem solvers and use multiple approaches throughout the test. If you always try to answer problems mathematically and conceptually, your score will suffer. If you always try to “game” problems by back-solving and number picking, your score will suffer. If you mix the two approaches and have all the tools at your disposal, your score will most definitely improve!

**SKILLBUILDER**
- Recognizing and transforming common algebraic equations
- Algebraic manipulation
Common Algebraic Equations Summary

The three common algebraic equations are particularly important for the GMAT. It is likely that you will be tested on them numerous times in any one test, so make sure you are fluent with all three. To summarize:

• There are two perfect squares that are important:
  1. \((x + y)^2 = x^2 + 2xy + y^2\)
  2. \((x – y)^2 = x^2 – 2xy + y^2\)

• Remember that you do not simply memorize these two equations. They are templates for anything in this form. Whenever you add together two things and square that expression, or subtract one thing from another and square that expression, this template applies. The template says that for the first example above you always square the first term, add double the product of the two terms, and add the square of the last term. For the second example above you always square the first term, subtract double the product of the two terms, and add the square of the last term.

Examples: \((2x + 5y)^2 = 4x^2 + 20xy + 25y^2\) \((3x – 4y)^2 = 9x^2 – 24xy + 16y^2\)

• You should be fluent enough with the perfect squares that you can move back and forth between forms quickly and without FOILing. As for all of the common algebraic equations, when they are presented in one form on the GMAT, you should recognize that form and be able to transform them quickly.

• The common mistake in working with the perfect squares is that people forget to double the product for the middle term. Always remember that the middle term is double the product of the two terms being added or subtracted.

• The difference of squares is heavily tested on the GMAT. Recognizing the difference of squares in any form is essential: \(x^2 – y^2 = (x + y)(x – y)\). Again, this is not one equation to memorize but a template for any time you have things in this form.

Examples: \(16x^2 – 1 = (4x + 1)(4x – 1)\) \((5x + 3y)(5x – 3y) = 25x^2 – 9y^2\)

• Whenever you are presented with one form of the difference of squares, transform it to the other form.
SECTION 5: INEQUALITIES

To this point we have dealt with expressions (mathematical values outside of an equation) and equations (a true statement in which one side is exactly equal to the other). The GMAT can also ask you about a third type of mathematical structure: an inequality. Inequalities take the following forms:

\[ a > b \]  
\[ a < b \]  
\[ a \geq b \]  
\[ a \leq b \]

\[ a > b \quad \text{a is greater than } b. \]
\[ a < b \quad \text{a is less than } b. \]
\[ a \geq b \quad \text{a is greater than or equal to } b. \]
\[ a \leq b \quad \text{a is less than or equal to } b. \]

**NOTE:** The following is also an inequality, but on the GMAT is used as a definition of a term, and is not something useful for calculation: \[ a \neq b \quad [a \text{ is not equal to } b]. \]

How to Manipulate Inequalities

The beauty of inequalities on the GMAT is that they allow you to do all of the same things that you can with equations:

\[ 2x + 5 > 7 \]
\[ 2x > 2 \quad \text{(subtract 5 from both sides)} \]
\[ x > 1 \quad \text{(divide both sides by 2)} \]

The only—but incredibly significant—difference is that you must flip the sign of the inequality whenever you multiply or divide by a negative!

Why? The demonstration on the next page should help. Let’s say that \( x > 5 \). That means that potential values of \( x \) include 6, 10, 15, 20, etc. If we were to multiply both sides by –1, we’d take those potential values of \( x \) and multiply them by –1, so potential values include –6, –10, –15, –20, etc. Those numbers are all less than –5. The absolute value—the distance away from 0—is the same, but the greater than/less than notion has flipped across the number line.
Inequalities can be particularly challenging when offered as part of a Data Sufficiency problem. Quite soon, if not already, you should have memorized that you must change the direction of the inequality when you multiply or divide by a negative. Where the GMAT can catch you napping, however, is in a case like the problem that you see on the next page.
17. Is \( x > 3y \)?

(1) \( \frac{x}{y} > 3 \)

(2) \( y > 0 \)

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient to answer the question asked.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient to answer the question asked.

(C) BOTH statements (1) and (2) TOGETHER are sufficient to answer the question asked, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient to answer the question asked.

(E) Statements (1) and (2) TOGETHER are NOT sufficient to answer the question asked, and additional data are needed.
LEARNING BY DOING

Be Careful Multiplying and Dividing

Did you catch the subtlety on that problem? Did statement 2 help tip you off? In statement 1, the temptation is to just multiply both sides by y to make the statement look just like the question:

\[ x > 3y \]

But without knowing whether y is positive or negative, we cannot multiply by y! The statement is just as likely to be \( x < 3y \), because we’d have to flip the sign if y were negative. (You can also try with numbers: x could be 4 and y could be 1, and x would be greater than 3y. But if x is –4 and y is –1, statement 1 is satisfied—but –4 is less than 3(–1).)

Statement 2 should clue you in. While it’s clearly not sufficient on its own, it ought to raise that red flag in your mind regarding statement 1: “Statement 2 says that y is positive. Wait—did I know that already when I said that statement 1 was sufficient? If not, I need to go back and reconsider: What if y were negative?” You need both statements together. The correct answer choice is C.
18. Is \( xy > 0? \)

(1) \( x - y > -5 \)

(2) \( x - 2y < -7 \)

(A) Statement 1 ALONE is sufficient, but statement (2) alone is not sufficient to answer the question asked.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient to answer the question asked.

(C) BOTH statements (1) and (2) TOGETHER are sufficient to answer the question asked, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient to answer the question asked.

(E) Statements (1) and (2) TOGETHER are NOT sufficient to answer the question asked, and additional data are needed.
LEARNING BY DOING
Combining Inequalities

How did you do this one? Did you think about conceptually? Number pick? Neither of those approaches is going to work well, if at all, on this problem. For most inequality problems, you have to manipulate the inequality algebraically and let the algebra tell the truth. In order to do that, you must be comfortable with rules for manipulation.

In equations with multiple variables, you have a choice of the substitution method or the elimination method. With inequalities, you can only use the elimination method, and to combine inequalities the signs must be facing the same way and you can only add; you can never subtract. After all, subtracting x is the same thing as saying: + (−x), and if you were to do that you’d be multiplying by −1. So when combining inequalities, always add.

For this problem, consider each statement alone. For each one, it is easily shown that x and y could each be positive, or one could be positive and one negative. Thus, this is quickly an answer choice C or E question. To make that choice between answer choices C and E, you must combine the inequalities to see what limits are put on x and y. To do that, you should first multiply the bottom inequality by −1 to get the signs facing the same way and eliminate x:

\[
\begin{align*}
x - y &> -5 \\
-1(x - 2y < -7) &\Rightarrow -x + 2y > 7 \\
\text{Let } y > 2
\end{align*}
\]

Then multiply the top one by −2 to eliminate y and learn the limits for x:

\[
\begin{align*}
-2(x - y > -5) &\Rightarrow -2x + 2y < 10 \\
-2x + 2y < 10 &\Rightarrow x - 2y < -7 \\
\text{Let } -x < 3 \quad \text{so } x > -3
\end{align*}
\]

Since y > 2 and x > −3, you do not know if the product will be positive or negative, so answer choice E is correct.

SKILLBUILDER

- Rules for manipulating inequalities
- General algebraic manipulation
19. Is \( x - y > r - s \)?

(1) \( x > r \) and \( y < s \)

(2) \( y = 2, s = 3, r = 5, \) and \( x = 6 \)

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient to answer the question asked.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient to answer the question asked.

(C) BOTH statements (1) and (2) TOGETHER are sufficient to answer the question asked, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient to answer the question asked.

(E) Statements (1) and (2) TOGETHER are NOT sufficient to answer the question asked, and additional data are needed.
LEARNING BY DOING
Inequalities and the Inconvenient Truth

On this question, it is easy to prove that statement 2 is sufficient, because values for each variable are given. For the first statement, you could pick a bunch of values (wondering if you picked enough and/or the right ones), or you could also consider it logically: If the number doing the subtracting on the left (x) is bigger than the one doing the subtracting on the right (r), and the number being subtracted on the left (y) is smaller than the one being subtracted on the right (s), then the distance between the numbers on the left must be larger than the right. But you might wonder about whether it matters where the numbers are on the number line (are some positive, some negative, etc.) and get confused. The simplest and fastest way to prove that statement 1 is sufficient is by algebraic manipulation (which people forget to do with inequalities).

Consider this data sufficiency drill problem, in which everyone understands that the first statement is sufficient:

Is \( x = y \)?

1. \( x - y = 0 \)

You would never think about this problem conceptually or number pick. You would just add y to both sides of the statement and say, "Yep. \( x = y \) and statement 1 is sufficient.

For statement 1 in the inequality problem (problem #19), it is really the same:

Is \( x - y > r - s \)?

1. \( x > r \) and \( y < s \)

Add the inequalities by getting the signs pointing the same way (multiply the second one by \(-1\))

\[
\begin{align*}
  x &> r \\
  -y &> -s \\
  x - y &> r - s
\end{align*}
\]

Therefore statement 1 is clearly sufficient. Answer choice D is correct.

SKILLBUILDER
- Rules for manipulating inequalities
- General algebraic manipulation
Inequalities with Absolute Value

While there are two approaches to these types of inequality problems, the most reliable is to create two separate inequalities that consider the two possible scenarios given by the absolute value sign (positive/zero and negative). Consider the following two examples:

\[ |x| < 5 \]

In this case, the absolute value of \( x \) is less than 5, and there are two possible scenarios: Either \( x \) is positive/zero or negative. If \( x \) is positive or zero, then the following case is true: \( x < 5 \). However, if \( x \) is negative, then the following inequality is true: \(-x < 5\) or, after manipulating by \(-1\), \( x > -5 \).

Taking them together, we know that \( x < 5 \) and \( x > -5 \). This can also be written as \(-5 < x < 5\).

Here is the visual representation of that on the number line:

```
-6  -5  -4  -3  -2  -1  0   1   2   3   4   5   6
```

\[ |x| > 5 \]

In this case the absolute value of \( x \) is greater than 5, and again there are two possible scenarios:

Either \( x \) is positive or negative. If \( x \) is positive, then the following case is true: \( x > 5 \). However, if \( x \) is negative, then the following inequality is true: \(-x > 5\) or, after manipulating by \(-1\), \( x < -5 \). Taking them together, we know that \( x > 5 \) or \( x < -5 \), which cannot be written as one statement. Here is the visual representation of that on the number line.

```
6    5    4    3    2    1    0   1   2   3   4   5   6
```

Summarized in simplistic terms, absolute values with inequalities can be considered in the following manner:

\[ |x| < y \text{ means } -y < x < y \]

\[ |x| > y \text{ means } x > y \text{ or } x < -y \]
Consider one last example where there is more than a variable within the absolute value sign:

\[ |x - 3| > 5 \]

Again we must consider the two scenarios: when \( x - 3 \) is zero or greater, and when \( x - 3 \) is negative. In the first scenario, we know that \( x - 3 > 5 \) and \( x > 8 \). However, when the value of \( x - 3 \) is negative, we must consider the second scenario, which is

\[ -(x - 3) > 5 \]

After simplifying this inequality, we have \( -x + 3 > 5 \) or \(-x > 2\) or \( x < -2 \). Putting these together, we know that \( x > 8 \) or \( x < -2 \).

While it is useful to understand conceptually what is going on, it is generally quicker to just use the template given on the previous page:

\[ \text{If } |x| > y \text{ means } x > y \text{ or } x < -y \]

From that you can see that \( |x - 3| > 5 \).

\( x - 3 > 5 \) or \( x - 3 < -5 \)

Simplifying, we see that \( x > 8 \) or \( x < -2 \).
Inequalities Summary

For most students, inequalities (particularly when presented in data sufficiency form) are a difficult content area. To succeed on inequality problems, you must first be clear about the following rules for manipulation:

- When adding to or subtracting from either side of an inequality, the same rules apply as when working with equations (do the same to both sides). You do not need to do anything to the inequality sign; you simply add or subtract the same thing from both sides.
- When multiplying or dividing with a positive number in an inequality, the same rules also apply as when working with equations (do the same to both sides). Simply multiply or divide all terms by the positive number, and you do not need to change the inequality sign.
- When multiplying or dividing with a negative number in an inequality, you must change the inequality sign. Simply multiply or divide all terms by the negative number, but then remember to switch the inequality sign. Why? Because for whatever is true on one side of 0 (for instance, if x>y) the opposite is true on the other side of zero (−x < −y).
- You cannot multiply or divide with variables in an inequality unless you know the sign of that variable. This is perhaps the most important rule to remember for harder inequality problems. If you do not know the underlying sign of a variable, then you do not know if you need to switch the sign or not.
- Inequalities, like equations, can be combined to eliminate variables. You can only combine inequalities when the inequality signs are facing in the same direction. If the signs are not facing the same way, you can multiply by a negative number to switch that sign.
- With absolute value, you must use the two-case approach to determine the two possibilities for the inequality. If there are multiple absolute values in an inequality, it may be too tedious (too many scenarios) to manipulate algebraically and you could use logic or number picking to solve.

The following points summarize the general strategic approach for dealing with inequality problems in data sufficiency form:

1. Attempt to manipulate algebraically as shown in the last three problems in this section. Using algebra will work on almost all Data Sufficiency inequality problems.

2. If you cannot manipulate algebraically (for instance, you can't multiply or divide because you don’t know the sign of the variable) or it is too cumbersome, try to use logic and your understanding of the number line.
3. If neither algebraic manipulation nor the number line is working, go to number picking, but make sure you are number picking with a purpose. Beware of the “black hole” of number picking with inequality problems in data sufficiency.
SECTION 6: YOU OUGHTA KNOW

Functions

What is a function? It is simply another way to write an algebraic expression with one variable. (The GMAT does not test multivariable functions.) Take the expression \( x^2 + 5 \). The function \( f \) for that expression can be denoted by the following: \( f(x) = x^2 + 5 \). In simple language, think of the input in this function as the value of \( x \), and the output of this function as the value defined by the expression \( x^2 + 5 \) for that value of \( x \). For this particular function, \( f(2) = 9 \) because \( 2^2 + 5 = 9 \).

Functions can be expressed with any letter, but on the GMAT it will generally be defined with \( f(x) \) or \( g(x) \). Let’s look at a couple more examples of functions:

\[
f(x) = \frac{x + 5}{\sqrt{x + 5}}
\]

For this function, \( f(4) = 3 \) because \( \frac{4 + 5}{\sqrt{4 + 5}} = \frac{9}{3} = 3 \).

**NOTE:** The domain of a function is defined as the set of all allowable inputs for the function. In this function, the domain is restricted by the square root and by the denominator. In most function problems, the domain is the set of all real numbers. Here, the domain is all values of \( x > -5 \) because the square root of a negative number is not a real number, and because a value of 0 in the denominator would be undefined.

Here is one more example:

\[
g(x) = \frac{1}{x} \quad (x + 5)
\]

For this function, \( g(8) = 26 \), because \( \frac{1}{8} (8 + 5) = 2(13) = 26 \).

**NOTE:** In this problem the domain of the function is all real numbers, because the output of this function will be a real number for any real number value of \( x \) (the input).
20. The function $f$ is defined by subtracting 25 from the square of a number and the function $g$ is defined as the square root of one-half of a number. If $g(f(x)) = 10$ then which of the following is a possible value of $x$?

(A) $-15$

(B) $-5$

(C) 0

(D) 5

(E) 25
LEARNING BY DOING
Functions Are Just Algebra

With a bit of practice, you should find that function problems are often just about “following directions,” but with the use of abstraction to make the directions just difficult enough to understand that the problem can be challenging. That is why it’s often helpful to construct your own conceptual understanding of what the function means, independent of the often-intimidating function notation. Here, the directions say to, for \( f(x) \), square \( x \) and subtract 25. Then you need to take whatever results there, and perform the \( g \) function, which is dividing by two, then taking the square root. So, mathematically, you have:

\[
g(x^2 - 25)
\]

or

\[
\sqrt{\frac{x^2 - 25}{2}} = 10
\]

Now it’s a matter of algebra.

Square both sides to eliminate the radical:

\[
\frac{x^2 - 25}{2} = 100
\]

Multiply out the denominator:

\[
x^2 - 25 = 200
\]

Add 25 to both sides

\[
x^2 = 225
\]

And note that \( x \) could be 15 or \(-15\). Only \(-15\) appears in the answer choices, so the answer must be answer choice A.

SKILLBUILDER
- Roots
- Quadratics
- Interpreting functions
Sequences

Sequences represent particular kinds of function problems on the GMAT. The domain of a sequence (its allowable input values) typically consists of the positive integers. The first term of a sequence is typically the output when the input is 1, the second term of a sequence is the output when the input is 2, etc.

The terms of the sequence are notated as $a_1$, $a_2$, $a_3$, …, $a_n$. In the sequence $a_n = n^2$, the first term ($a_1$) is 1, the second term ($a_2$) is 4, and so on, such that the sequence is: 1, 4, 9, 16, 25, 36, ….

21. The sequence $a_1$, $a_2$, $a_3$, …, $a_n$ is defined by $a_n = a_{n-2} + a_{n-1}$ for all $n > 2$. If $a_3 = 2$ and $a_5 = 5$, what is the value of $a_6$?

(A) 7  
(B) 8  
(C) 10  
(D) 12  
(E) 13
LEARNING BY DOING
Understand Sequence Denotation

Like function problems, sequence problems are often most difficult simply due to their use of abstraction. The notation for these problems is awkward and intimidating to most. It’s helpful to develop a general understanding of what’s happening in the problem; turn algebraic notation into your own conceptual understanding. Here, the sequence is essentially “each term is the sum of the previous two terms.” So if the fifth term is 5 and the third term is 2, then the fourth term must be 3, because 5 must equal 2 + the fourth term. And if the fourth term is 3 and the fifth term is 5, then the sixth term (the sum of terms four and five) is 8, answer choice B.

THINK LIKE THE TESTMAKER
Abstraction

Don’t let the difficult denotation in function and sequence problems confuse you. Establish the function and/or sequence in your mind either conceptually or by using a few numbers as examples. Once the function or pattern is established, these problems are generally not that difficult. It is essential that you practice several of these problems so that you become proficient at interpreting the difficult denotation.

SKILLBUILDER
• Interpreting sequences
• General algebraic manipulation
Properties of Zero

Many a GMAT problem can turn on you because of the presence of the nasty number 0, an easy number to overlook, because 0 literally means “nothing.” It’s like it’s not even there! The number 0 may well be the greatest and most versatile weapon that the GMAT has to work with against you. Let’s take a pop quiz about the number 0.

Zero Drill

Is 0:

- Even, odd, or neither?
- Positive, negative, or neither?
- A multiple of 3 or not?
Drill Solutions

0 is even, as it is evenly divisible by 2.

0 is neither positive nor negative. In fact, it’s the dividing line between the two (“positive” means “greater than 0”; “negative” means “less than 0”). This can be critical, as the words “positive” and “non-negative” have a slight but significant difference: Positive excludes 0, but non-negative includes 0. So a question that involves the word “non-negative” is often giving you a clue: Check for 0!

0 is a multiple of 3. In fact, it is a multiple of every number, because 0(x) = 0 for all values of x.
Now let’s look at a GMAT problem that paints 0 in a different light. It can be a dangerous weapon for the GMAT to use against you, but it’s also an asset for you when it comes to certain exponent problems.

22. If \(3^x4^y = 177,147\) and \(x – y = 11\), then \(x =\)

(A) Undefined
(B) 0
(C) 11
(D) 177,136
(E) 177,158
LEARNING BY DOING

0 as an Exponent

How does 0 factor into this problem? Among the only things you know about 177,147 is that it is not even, which makes the $4^y$ term on the left side of the equation problematic. This number cannot be divisible by 4. How do you avoid that? Remember: Anything to the 0 power is 1, so if $y$ were to equal 0, then the problem is just $3^x = 177,147$. That is permissible; if $y$ is 0, then $x$ must be 11. The correct answer choice is C.

So, yes, it’s an algebraic rule that $x^0 = 1$ (even when $x$ is 0—although there is still a bit of debate between mathematicians about whether that should expressly be the case), but have you ever wondered why? Although mathematical rules may seem arbitrary, they almost never are; there’s always a way to prove them, and you can make yourself an excellent test-taker by trying to prove them to yourself.

In this case, if you accept the rules that $x^{-y} = \frac{1}{x^y}$ and that $x^y \cdot x^z = x^{(y+z)}$, then there are two ways to express the statement $x^2 \cdot x^{-2}$:

\[ x^{(2-2)} = x^0 \]

and

\[ \frac{x^2}{x^2} = 1 \]

Therefore, $x^0 = 1$. 

SKILLBUILDER

• Exponent rules
• Odd/even properties
Difficult Rationalizing of Denominators

The GMAT generally does not include in its answer choices any mathematical expressions in which a radical ($\sqrt{x}$) appears in the denominator of a fraction. Accordingly, if your calculations yield a radical in the denominator, you should be prepared to take one last step: rationalize the denominator.

Should the radical only include one term, the standard method to remove the radical is to multiply by 1, strategically, so that you multiply by that radical over itself:

$$\frac{3}{\sqrt{2}} \Rightarrow \text{Multiply by } \frac{\sqrt{2}}{\sqrt{2}} \text{ to get } \frac{3\sqrt{2}}{2}.$$ 

If the radical includes two terms, either added or subtracted, your job becomes significantly more difficult. Fortunately, there's a method (We hesitate to say “trick” — but it's tricky if you haven't seen it.)

23. What is the value of $\frac{2}{2 - \sqrt{2}}$?

(A) 1

(B) 2

(C) $1 + \sqrt{2}$

(D) $2 + \sqrt{2}$

(E) 4
LEARNING BY DOING

Difference of Squares: The Great Transformer

In this problem, the multi-part denominator poses an algebraic challenge. Try to multiply both the numerator and denominator by the denominator, and you will still have a messy radical in the denominator. Algebraically, the solution is to employ the difference of squares:

\[ \frac{2}{2 - \sqrt{2}} \cdot \frac{2 + \sqrt{2}}{2 + \sqrt{2}} \]

Notice that by adding the square root of 2 in the denominator, the difference of squares rule (you could FOIL this, but that’s time-consuming) will drop the \( \sqrt{2} \) terms, leaving just \( 4 - 2 \) (which equals 2) in the denominator:

\[ \frac{2(2 + \sqrt{2})}{2^2 - (\sqrt{2})^2} = \frac{2 + \sqrt{2}}{2} \quad \rightarrow \text{Answer choice D is correct.} \]

When rationalizing denominators with addition or subtraction in them, the difference of squares rule is a nearly magical way to transform the expression. Accordingly, the difference of squares rule is one you should commit to memory and keep top-of-mind on test day.

SKILLMEET STRATEGY

The Answer Choices Are Part of the Problem

While it is easiest to solve this problem with algebra, what if didn’t register with you to use the difference of squares? Good problem solvers will find another way! For instance, you could solve the problem through a mix of estimation and using answer choices—a tool that can be helpful for many different problems. If you recognize that the square root of 2 is in the 1.4 range, then \( \frac{2}{2 - \sqrt{2}} \) is 2 divided by around 0.6. 2 divided by 0.5 is 4, so this should be a bit less than that, and only choice D is in that ballpark. As you will continue to see throughout the quantitative lessons, the answer choices on problem-solving questions are assets, not just choices. In this case, the answer choices give you an opportunity to solve via estimation, or they tip you off that you must find a way to rationalize the denominator (as none of the answer choices include a denominator). Let the answer choices guide your effort when they provide clues such as these.

SKILLBUILDER

- Common algebraic equations
- Roots
Algebraic Number Properties

In addition to the number properties that you learned in the Arithmetic book, there are several important attributes of numbers that relate specifically to the algebra material. Most of these relate to exponents and roots, and are just offshoots of the positive/negative properties that you learned in Arithmetic (which is why we have put them in the “You Oughta Know” section).

Number Properties Relating to Exponents and Roots

The most important algebraic number properties relate to the positive and negative attributes of numbers with exponents and roots. You learned that all numbers have two square roots but that, when the radical sign is used, only the principal square root is considered. To review:

\[ \sqrt{16} = \text{positive 4 only, but when } x^2 = 16 \text{ then } x = +4 \text{ and } -4 \]

Exponent Number Properties

**NOTE:** For all examples below, zero is always a possibility, but we have given the properties assuming non-zero values.

1. Any number or variable raised to an even exponent will always be positive.
   Examples: \( x^2 \quad (-4)^6 \quad y^{20} \)

2. When a variable is raised to an even exponent, we know the result is positive, but we do not know the sign of the variable.
   Examples: If \( x^2 = 25 \), then \( x = 5 \) and \(-5\)  \( (5 - x)^4 = 16 \)  \( x = 3 \) and 7

3. When a number or variable is raised to an odd exponent, the result can be positive or negative, depending on the sign of the base.
   Examples: \( x^3 \), \( y^5 \), or \( x^{77} \) can be either + or –  \( (-4)^3 = -64 \)  \( 5^3 = 125 \)

4. When a variable is raised to an odd exponent, the sign of the result determines the sign of the variable.
   Examples: If \( x^3 = -125 \), then \( x = -5 \)  If \( x^3 = 64 \), then \( x = +4 \)

5. When a number between 0 and 1 is squared, the result becomes smaller. For all other real numbers greater than 1 or less than 0, the square of that number becomes greater.
   Examples: \( \left( \frac{1}{2} \right)^2 = \frac{1}{4} \)  \( .4^2 = .16 \)  \( -( .4)^2 = -( .16) \)
Root Number Properties

As roots can all be expressed as exponents, the same thought processes relate to positive/negative properties relating to roots:

1. For even roots of all positive numbers, there are two solutions—one positive and one negative. However, when the radical sign is used (\(\sqrt{\phantom{0}}\)), the question is only asking for the principal square root. There is only one solution for an even root of 0, and the even root of a negative number is not a real number, so there is no solution on the GMAT.

Examples: The two roots of 16 are +4 and –4, but \(\sqrt{16} = +4\) only. \(\sqrt{0} = 0\). \(\sqrt{-16}\) is not a real number so there is no solution on the GMAT for this calculation.

2. For odd roots of all real numbers, there is exactly one solution, which can be negative, positive, or 0.

Examples: \(\sqrt[3]{-27} = -3\) \(\sqrt[5]{32} = 2\) \(\sqrt[3]{0} = 0\)

3. When taking the square root of number between 0 and 1, the result is greater than the original number. For all positive numbers greater than 1, the square root of that number will be less than the original number.

Examples: \(\sqrt{\frac{1}{4}} = \frac{1}{2}\) \(\sqrt{25} = 5\)
24. Is $xy < 0$?

(1) $x^2y^3 < 0$

(2) $xy^2 > 0$

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient to answer the question asked.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient to answer the question asked.

(C) BOTH statements (1) and (2) TOGETHER are sufficient to answer the question asked, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient to answer the question asked.

(E) Statements (1) and (2) TOGETHER are NOT sufficient to answer the question asked, and additional data are needed.
LEARNING BY DOING
Positive/Negative Properties with Exponents

To answer the question definitively, you must be able to determine the sign of x and y with certainty. In statement 1, you learn that the product of $x^2$ and $y^3$ is negative. Because $x^2$ is always positive, it is clear that $y^3$ must be negative. Furthermore, if $y^3$ is negative, then y must be negative. However, while you know that $x^2$ is positive, you cannot determine with any certainty the sign of x. Therefore, from statement 1, you learn x could be positive or negative, and y is negative. That is not sufficient to answer the question. Similarly, in statement 2 you learn that the product of x and $y^2$ is positive. Because $y^2$ is positive, x must be positive. Again, however, y could be either positive or negative, so statement 2 is not sufficient. Taking the statements together, it is certain that y is negative and x is positive, so it is possible to definitively answer the question. The correct answer choice is C.
Proper Fractions with Exponents

Another important set of number properties relates to what happens when you square proper fractions (fractions between 0 and 1). Like the previous positive/negative properties, this is a simply a more complicated version of an important arithmetic number property relating to fractions. Consider the following example, which tests exactly one of these number properties:

25. If $x^2 < x$, which of the following could be a value of $x$?

(A) $-6$
(B) $-\frac{2}{3}$
(C) $\frac{6}{7}$
(D) $\frac{5}{4}$
(E) 25
LEARNING BY DOING
Fractions Create Unique Results with Exponents

Normally, when you square a number, it gets bigger. You should ask yourself how it could get smaller, because you are told that $x^2$ is less than $x$ in the question stem. Either using the answer choices or understanding it abstractly, you should realize that this is only possible if $x$ is a positive proper fraction (which means between 0 and 1). Answer choice C is the only choice meeting those conditions. Any negative number is eliminated, because its square is always positive and thus greater than itself. Any number greater than 1 is eliminated, because its square will result in a number greater than itself.

SKILLBUILDER
- Arithmetic number properties
- Fractions
You Oughta Know Summary

The information in this section is fairly important for the GMAT. Below is summary of important information for each component:

1. **Functions**
   - Functions are simply another way to present algebraic equations. To succeed on function questions, make sure that you understand the denotation and know how to substitute into that function.
   - Property of function questions are particularly confusing and are presented with several examples in the homework problems.

2. **Sequences**
   - Sequences are relatively common on the GMAT and have more complicated denotation than functions.
   - Understand that the subscript in sequence denotation represents the number term and that the nature of the sequence is always defined in the question stem. Your ability to translate that definition of the sequence into comprehensible language is essential. For instance, if you are given $a_n = a_{n-2} + a_{n-1}$ for all values of $n > 2$, you must translate that to mean that each term in the sequence is equal to the sum of the preceding two terms.

3. **Properties of Zero**
   - Zero is a particularly important number for the GMAT, as it gives unique results in many mathematical scenarios, particularly when 0 is used as an exponent in algebra problems.
   - Zero is an even number that is neither negative nor positive.

4. **Rationalizing Denominators**
   - Rationalizing denominators when you must employ the difference of squares is particularly difficult. Remember that because the difference of squares contains no middle term, anticipating such a calculation is important for rationalizing tricky denominators.

5. **Algebraic Number Properties with Roots and Exponents**
   - Positive/negative number properties are much more confusing when exponents are involved. Make sure you understand that numbers raised to even powers can only be positive or 0.
   - When positive proper fractions are squared, they get smaller.
   - Root number properties are also important but less commonly tested.
Algebraic Calculations

26. If \((b - 3)(4 + \frac{2}{b}) = 0\) and \(b \neq 3\), then \(b =\)

(A) \(-8\)
(B) \(-2\)
(C) \(-\frac{1}{2}\)
(D) \(\frac{1}{2}\)
(E) 2
27. If \( x = 5 - 4k \) and \( y = 5k - 3 \), then for what value of \( k \) does \( x = y \)?

(A) 0
(B) \( \frac{8}{9} \)
(C) \( \frac{9}{8} \)
(D) 2
(E) 8
28. If $x$, $y$, and $z$ are all nonzero numbers, and $x = y + z$, which of the following is equal to 1?

(A) $\frac{y - z}{x}$

(B) $\frac{y - x}{z}$

(C) $\frac{z - x}{y}$

(D) $\frac{z - y}{x}$

(E) $\frac{x - z}{y}$
29. In the expression above, if \( xn \neq 0 \), what is the value of \( S \)?

(1) \( x = 2n \)

(2) \( n = \frac{1}{2} \)

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.
30. If $5x = 4y$, which of the following is NOT true?

(A) $\frac{x + y}{y} = \frac{9}{5}$

(B) $\frac{y}{y - x} = 5$

(C) $\frac{x - y}{y} = \frac{1}{5}$

(D) $\frac{4x}{5y} = \frac{16}{25}$

(E) $\frac{x + 3y}{x} = \frac{19}{4}$
31. If \( \frac{x}{x+y} = 6 \), then \( \frac{y}{y+x} = \)

(A) \(-5\)

(B) \(\frac{5}{11}\)

(C) 1

(D) \(\frac{11}{5}\)

(E) 5
32. \( \frac{2(x - 2)}{5} + \frac{24 - 12x}{6} + 2x = \frac{4x}{5} + 3 \). Solve for \( x \).

(A) 0
(B) 0.5
(C) 1
(D) 1.5
(E) 2
33. If \( x \neq 3 \) and \( \frac{x^2 - 9}{2y} = \frac{x - 3}{4} \), then in terms of \( y, x = \)

(A) \( \frac{y - 6}{2} \)

(B) \( \frac{y - 3}{2} \)

(C) \( y - 3 \)

(D) \( y - 6 \)

(E) \( \frac{y + 6}{2} \)
34. If x, y, and z are positive, is \( x = \frac{y}{z^2} \)?

(1) \( z = \frac{y}{xz} \)
(2) \( z = \sqrt{\frac{y}{x}} \)

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.
(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.
(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.
(D) EACH statement ALONE is sufficient.
(E) Statements (1) and (2) TOGETHER are NOT sufficient.
35. Is $2^x$ greater than 100?

(1) $2^{\sqrt{x}} = 8$

(2) $\frac{1}{2^x} < 0.01$

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.
36. If \( k \neq 0 \) and \( \frac{z^2}{k} + 4z + 3 = \frac{z}{k} \), then \( k = \)

(A) \( \frac{z^2 - 3}{4} \)

(B) \( \frac{-z^2 - 4z}{3} \)

(C) \( -3z(z + 4) \)

(D) \( \frac{z - z^2}{4z + 3} \)

(E) \( z^2 + 4z - 3 \)
37. \((\sqrt{11} + \sqrt{11} + \sqrt{11})^2 = \)

(A) 363  
(B) 121  
(C) 99  
(D) 66  
(E) 33
38. If \( \frac{4-x}{2+x} = x \), what is the value of \( x^2 + 3x - 4 \)?

(A) \(-4\)
(B) \(-1\)
(C) 0
(D) 1
(E) 2
Algebraic Word Problems

39. How many hours does it take Jennifer to run $y$ miles if she runs at a speed of $x$ miles per hour?

(A) $\frac{x}{y}$
(B) $\frac{y}{x}$
(C) $xy$
(D) $\frac{60x}{y}$
(E) $\frac{y}{60x}$
40. Missy ate \( m \) more crackers than Audrey did, from a box that originally contained \( n \) crackers. Together, they finished the box. Which of the following represents the number of crackers that Missy ate?

(A) \( \frac{n + m}{2n} \)
(B) \( \frac{m - n}{2} \)
(C) \( \frac{n - m}{2} \)
(D) \( \frac{m + n}{2} \)
(E) \( \frac{n}{2 + m} \)
41. The sum of the heights of two high-rises is \( x \) feet. If the first high-rise is 37 feet taller than the second, how tall will the second high-rise be after they add an antenna with a height of \( z \) feet to the top?

(A) \( \frac{x + z}{2 + 37} \)

(B) \( 2x - (37 + z) \)

(C) \( \frac{x - 37}{2} + z \)

(D) \( \frac{x}{2} - 37 + z \)

(E) \( \frac{2x - 37}{z} \)
42. A number $x$ is multiplied by 3, and this product is then divided by 5. If the positive square root of the result of these two operations equals $x$, what is the value of $x$ if $x \neq 0$?

(A) $\frac{25}{9}$

(B) $\frac{9}{5}$

(C) $\frac{5}{3}$

(D) $\frac{3}{5}$

(E) $\frac{9}{25}$
There are 1,600 jelly beans divided between two jars (x and y). If there are 100 fewer jelly beans in jar x than three times the number of beans in jar y, how many beans are in jar x?

(A) 375  
(B) 950  
(C) 1,150  
(D) 1,175  
(E) 1,350
Exponents and Roots

44. If $m$ is an integer such that $(-2)^{2m} = 2^{9-m}$, then $m =$

(A) 1
(B) 2
(C) 3
(D) 4
(E) 6
45. If x and y are positive integers and \( x^y = x^{2y} - 3 \), what is the value of \( x^y \)?

(1) \( x = 2 \)

(2) \( x^3 = 8 \)

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.
46. If \(\frac{0.0024 \cdot 10^j}{0.08 \cdot 10^r} = 3 \cdot 10^6\), then \(j - r\) equals which of the following?

(A) 9  
(B) 8  
(C) 7  
(D) 6  
(E) 5
47. \( \frac{(0.08)^{-4}}{(0.04)^{-3}} = \)

(A) \( \frac{-16}{25} \)

(B) \( \frac{16}{25} \)

(C) \( \frac{25}{16} \)

(D) 16

(E) 25
\[
\frac{x^{-5}(x^4)^2(x^3)}{x(x^{-7})(x^{-3})^4}
\]

48. The above equation may be rewritten as which of the following?

(A) \( \frac{1}{x^{25}} \)

(B) \( \frac{1}{x^8} \)

(C) \( x^6 \)

(D) \( x^{20} \)

(E) \( x^{32} \)
49. If \((2^x)(2^y) = 8\) and \((9^x)(3^y) = 81\), then \((x, y)\) equals which of the following?

(A) \((1, 2)\)

(B) \((2, 1)\)

(C) \((1, 1)\)

(D) \((2, 2)\)

(E) \((1, 3)\)
50. If $x$ is a positive integer, what is \( \left( \frac{2^x}{2^{-x}} \right)^x \)?

(A) 1
(B) \(2^{2x^2}\)
(C) \(2^{(2x)^2}\)
(D) \(4^{2x}\)
(E) \(4^{(2x)^2}\)
51. The size of diamonds is measured in carats. If the price of a diamond doubles for every 0.5 carats, which of the following is worth the most?

(A) One 4-carat diamond
(B) Fourteen 1-carat diamonds
(C) Four 2-carat diamonds and eight 1-carat diamonds
(D) One 1-carat diamond, two 2-carat diamonds, and three 3-carat diamonds
(E) Three 1.5-carat diamonds and three 2.5-carat diamonds
52. The variable \( x \) is inversely proportional to the square of the variable \( y \). If \( y \) is divided by \( 3a \), then \( x \) is multiplied by which of the following?

(A) \( \frac{1}{9a} \)

(B) \( \frac{1}{9a^2} \)

(C) \( \frac{1}{3a} \)

(D) \( 9a \)

(E) \( 9a^2 \)
53. If \( m \) and \( n \) are integers and \( \frac{36}{3^m} = \frac{1}{3^m} + \frac{1}{3^n} \), what is the value of \( m + n \)?

(A) −2
(B) 0
(C) 2
(D) 3
(E) 5
54. \[ \frac{15}{2^{-5} + 2^{-6} + 2^{-7} + 4^{-4}} = \]

(A) \(2^7\)
(B) \(2^8\)
(C) \(2^9\)
(D) \(15(2^7)\)
(E) \(15(2^8)\)
55. What is the value of \( \left( \frac{3}{\sqrt[3]{512}} \right)^2 \)?

(A) \( \frac{1}{16} \)

(B) \( \frac{1}{8} \)

(C) \( \frac{1}{2} \)

(D) 2

(E) 8
56. If \( 5 \cdot \sqrt{125} = \frac{1}{5^x} \), then \( x = \)

(A) \(-4\)
(B) \(\frac{-1}{\sqrt{2}}\)
(C) 0
(D) \(\frac{1}{\sqrt{2}}\)
(E) 1
57. If a and b are integers and \( ab - a \) is odd, which of the following must be odd?

(A) \( b^2 \)

(B) \( b \)

(C) \( a^2 + b \)

(D) \( ab \)

(E) \( ab + b \)
58. Which of the following equations has a root in common with 

\[ x^2 - 7x + 12 = 0 \]?

(A) \[ x^2 + 1 = 0 \]

(B) \[ x^2 - x - 12 = 0 \]

(C) \[ x^2 - 8x - 4 = 0 \]

(D) \[ 3x^2 - 9 = 0 \]

(E) \[ x^2 + x - 6 = 0 \]
59. Is $x > 0$?

(1) $x^2 = 9x$

(2) $x^2 = 81$

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.
60. For how many integers $n$ is $n + n = n \cdot n$?

(A) None

(B) One

(C) Two

(D) Three

(E) More than three
Common Algebraic Equations

61. Is \( x^2 - y^2 \) a positive number?

(1) \( x - y \) is a positive number.

(2) \( x + y \) is a positive number.

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.
62. If $4x^2 + 9y^2 = 100$ and $(2x + 3y)^2 = 150$, then what is the value of $6xy$?

(A) $5(2 + \sqrt{6})$

(B) $10\sqrt{6}$

(C) 25

(D) 50

(E) 100
63. $0 < x < y$, and $x$ and $y$ are consecutive integers. If the difference between $x^2$ and $y^2$ is 12,201, then what is the value of $x$?

(A) 6,100
(B) 6,101
(C) 12,200
(D) 12,201
(E) 24,402
64. Which of the following equations is NOT equivalent to $25x^2 = y^2 - 4$?

(A) $25x^2 + 4 = y^2$
(B) $75x^2 = 3y^2 - 12$
(C) $25x^2 = (y + 2)(y - 2)$
(D) $5x = y - 2$
(E) $x^2 = \frac{y^2 - 4}{25}$
65. If \( x \) is positive, then \( \frac{1}{\sqrt{x+1} + \sqrt{x}} = \)

(A) 1
(B) \( x \)
(C) \( \frac{1}{x} \)
(D) \( \sqrt{x+1} - \sqrt{x} \)
(E) \( \sqrt{x+1} + \sqrt{x} \)
66. If \( x \neq 0 \) and \( x = \sqrt{4xy - 4y^2} \), then in terms of \( y \), \( x = \)

(A) 2y  
(B) y  
(C) \( \frac{y}{2} \)  
(D) \( \frac{4y^2}{1 - 4y} \)  
(E) -2y
67. The expression \((\sqrt{8 + \sqrt{63}} + \sqrt{8 - \sqrt{63}})^2 =\)

(A) 20
(B) 19
(C) 18
(D) 17
(E) 16
Algebraic Number Properties

68. Is $x > y$?

(1) $x^2 > y^2$

(2) $x - y > 0$

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.
69. Is \( ab < 0 \)?

(1) \( a^4 b^9 c^2 < 0 \)

(2) \( a(bc)^6 > 0 \)

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.
70. What is the units digit of $9^{19} - 7^{15}$?

(A) 2
(B) 4
(C) 5
(D) 6
(E) 7
71. What is the units digit of \((23)^6(17)^3(61)^9\)?

(A) 1
(B) 3
(C) 5
(D) 7
(E) 9
72. If \( x^2 + y^2 = 100 \), \( x \geq 0 \), and \( y \geq 0 \), the maximum value of \( x + y \) must be which of the following?

(A) Less than 10
(B) Greater than or equal to 10 and less than 14
(C) Greater than 14 and less than 19
(D) Greater than 19 and less than 23
(E) Greater than 23
73. What is $\sqrt{x^2y^2}$ if $x < 0$ and $y > 0$?

(A) $-xy$
(B) $xy$
(C) $-|xy|$
(D) $|y|x$
(E) No solution
74. If x and y are nonnegative integers, what is the value of y?

(1) \(3^x = 5^y\)

(2) \(|y| = -y\)

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.
75. What is the value of y?

   (1)  \( x^2 - y^2 = 5 \)

   (2)  \( x \) and \( y \) are each positive integers.

   (A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

   (B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

   (C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

   (D) EACH statement ALONE is sufficient.

   (E) Statements (1) and (2) TOGETHER are NOT sufficient.
76. The sum of the digits of \((10^n)^y - 64\) is 279. What is the value of \(xy\)?

(A) 28
(B) 29
(C) 30
(D) 31
(E) 32
Inequalities

77. Which of the following describes all values of \( n \) for which \( n^2 - 1 \geq 0 \)?

(A) \( n \geq 1 \)
(B) \( n \leq -1 \)
(C) \( 0 \leq n \leq 1 \)
(D) \( n \leq -1 \) or \( n \geq 1 \)
(E) \( -1 \leq n \leq 1 \)
78. If \(-2 \leq m \leq 0\) and \(n > 19\), which of the following CANNOT be the value of \(mn\)?

(A) \(-48\)
(B) \(-38\)
(C) \(-20\)
(D) 0
(E) 19
79. If \( x \) is an integer, what is the value of \( x \)?

(1) \(-2(x + 5) < -1\)

(2) \(-3x > 9\)

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.
80. If \( b < 2 \) and \( 2x - 3b = 0 \), which of the following must be true?

(A) \( x > -3 \)

(B) \( x < 2 \)

(C) \( x = 3 \)

(D) \( x < 3 \)

(E) \( x > 3 \)
81. Is \( x^2 + y^2 > 100? \)

(1) \( 2xy < 100 \)

(2) \( (x + y)^2 > 200 \)

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.
82. Is \( xy > 0 \)?

(1) \( x - y > -5 \)

(2) \( x - 2y < -7 \)

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.
83. If \( x \) and \( y \) are integers and \( y = |x + 3| + |4 - x| \), does \( y \) equal 7?

(1) \( x < 4 \)

(2) \( x > -3 \)

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.
84. Which of the following inequalities is equivalent to \( x > -4 \)?

(A) \(-5x + 3 < 15 - 2x\)

(B) \(1.75x - 4 < 0.25x - 10\)

(C) \(-2x + 2 < 2(x - 2) - 2x - 2\)

(D) \(4(x - 4) < 10(4 - x)\)

(E) None of the above
85. If \( y \neq 0 \) and \( y \neq -1 \), which is greater, \( \frac{x}{y} \) or \( \frac{x}{y + 1} \)?

(1) \( x \neq 0 \)

(2) \( x > y \)

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.
86. If \(-1 < x < 1\) and \(x \neq 0\), which of the following inequalities must be true?

I. \(x^3 < x\)
II. \(x^2 < |x|\)
III. \(x^4 - x^3 > x^3 - x^2\)

(A) I only
(B) II only
(C) III only
(D) II and III only
(E) I, II, and III
Sequences and Functions

87. If the operation € is defined for all x and y by the equation \( x \text{ € } y = 2xy \), then 3 € (4 € 5) =

(A) 80  
(B) 120  
(C) 160  
(D) 240  
(E) 360
88. For all numbers \( q \) and \( r \), the operation \( \Theta \) is defined by \( q \Theta r = (q + 3)(r - 2) \). If \( 1 \Theta z = -8 \), then \( z \) is equal to which of the following?

(A) \(-3\)

(B) \(-2\)

(C) \(-1\)

(D) \(0\)

(E) \(2\)
89. Let \( f(n) \) be the number of distinct factors \( n \) has. For example, \( f(20) = 6 \), because 20 has six factors (1, 2, 4, 5, 10, and 20). Which of the following products is equal to 225?

(A) \( f(10) \cdot f(100) \)

(B) \( f(100) \cdot f(1,000) \)

(C) \( f(1,000) \cdot f(10,000) \)

(D) \( f(100) \cdot f(10,000) \)

(E) \( f(10) \cdot f(1,000) \)
90. In the infinite sequence \( a_1, a_2, a_3, \ldots, a_n = n^2 \). What is \( a_{1,323} - a_{1,322} \)?

(A) 2,245  
(B) 2,645  
(C) 5,290  
(D) 5,545  
(E) 5,790
91. If $g$ is a function defined for all $x$ by $g(x) = \frac{x^4}{16}$, then what is the value of $g(2x)$ in terms of $g(x)$?

(A) $\frac{g(x)}{16}$

(B) $\frac{g(x)}{4}$

(C) $4g(x)$

(D) $8g(x)$

(E) $16g(x)$
92. For which of the following functions is \( g(c - d) = g(c) - g(d) \) for all positive numbers \( c \) and \( d \)?

(A) \( g(x) = x^3 \)

(B) \( g(x) = x + 5 \)

(C) \( g(x) = \sqrt{3x} \)

(D) \( g(x) = 5x \)

(E) \( g(x) = \frac{15}{x} \)
93. In the sequence 1, 2, 2, ..., \(a_n\), ..., \(a_n = a_{n-1} \cdot a_{n-2}\). The value of \(a_{13}\) is how many times the value of \(a_{11}\)?

(A) 2  
(B) \(2^3\)  
(C) \(2^{32}\)  
(D) \(2^{64}\)  
(E) \(2^{89}\)
94. The infinite sequence \( a_1, a_2, \ldots, a_n, \ldots \) is such that \( a_1 = 7, a_2 = 8, a_3 = 10 \), and 
\[ a_n = a_{n-3} + 7 \]
for values of \( n > 3 \). What is the remainder when \( a_n \) is divided by 7?

(1) \( n \) is a multiple of 3.

(2) \( n \) is an even number.

(A) Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

(B) Statement (2) ALONE is sufficient, but statement (1) alone is not sufficient.

(C) BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

(D) EACH statement ALONE is sufficient.

(E) Statements (1) and (2) TOGETHER are NOT sufficient.
ANSWER KEY

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